

## **Identifying Monetary Policy Shocks**

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### **In Recent Developments in SVAR**

- 1 Modest Policy Changes: Leeper and Zah (2003)
- 2 Factor-Augmented VAR (FAVAR)
- 3 Lucas Critique and Lucas Programme

## 1 Introduction

Source: Christiano et al. (1998), Favero (2001, 6.3)

### 1.1 Outline

On the methodical side, the first two lectures are intended to familiarize you with the method of vector autoregressions (VAR). While the first lecture will introduce to basic concepts (VAR, SVAR), the second lecture presents a recent and more sophisticated development, the so called factor-augmented VAR (FAVAR). On the macroeconomic side, the lecture is about the macroeconomic effects of monetary policy. More precisely, we will discuss how to identify monetary policy shocks (a concept which is introduced in section 1.2) and about how to capture the effects of these shocks on economic variables. In this context, VARs have two different applications. First, VARs can be directly applied to identify and assess the effects of monetary policy shocks in a stable institutional environment. Second, VARs can be used to derive stylized facts about the response of the economy to monetary policy shocks. These stylized facts can then be further used to evaluate structural macroeconomic models that forecast the effects of monetary policy under changing monetary policy rules and institutions. This second application of VARs belongs to the so called Lucas-programme, which will be presented in the second lecture.

### 1.2 Monetary Policy Shocks

The idea that monetary policy systematically depends on economic developments can be captured in a monetary policy feedback rule. Such a rule links (systematic) monetary policy actions  $S_t$  to a set of variables  $I_t$  that characterize the state of the economy:

$$S_t = f(I_t) + \sigma_s \varepsilon_t^s \quad (1)$$

where  $f(\cdot)$  is usually supposed to be a linear function. The error term  $\varepsilon_t^s$  is called a (exogenous) monetary policy shock. Here we normalize the monetary policy shock to have unit variance, while  $\sigma_s$  is the standard deviation of the term  $\sigma_s \varepsilon_t^s$ .

But why not considering any change monetary authority's policy instrument  $S_t$  as a monetary policy shock? This would be misleading because monetary policy makers respond to various monetary and nonmonetary developments in the economy. In other words, monetary policy action  $S_t$  will depend on the effect of various shocks to an economy. Thus the effects of monetary policy  $S_t$  can not be isolated, but are always related to economic developments that can not be compared over time and across countries.

Christiano et al. (1998) suggest three possible interpretations of the monetary policy shock  $\varepsilon_t^s$ . First, the error term may reflect exogenous shocks to preferences of the monetary authority, possibly induced by institutional or personal changes in the authority. Second, the error term could reflect technical factors, such as measurement errors in the information set available to the monetary authority at the time of decision-making. Third, the error term could reflect exogenous variation in monetary policy caused by strategic considerations. Christiano et al.

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(1998, 8) refer to studies that present frameworks in which the monetary authority strategically reacts to the expectations of private agents, which may lead to exogenous monetary policy shocks captured in  $\varepsilon_t^s$ .

## 2 Identifying Monetary Policy Shocks

### 2.1 Estimation of a Feedback Rule

*Source: Andersen and Jordan (1968), Sims (1980), Enders (2004, 5.4), Favero (2001, 3.2-3)*

#### 2.1.1 Structural Macroeconomic Models: Cowles Commission Approach

The traditional approach, usually referred to as "Cowles Commission" approach, assesses the effects of monetary policy shocks in the framework of large structural equation models. Essentially a structural model of this tradition relates a (k)-dimensional vector  $Y$  of endogenous variables to its own lagged realisations as well as to contemporaneous and lagged values of exogenous variables contained in the (k+j)-dimensional vector  $X$ :

$$\begin{aligned} A_0 Y_t &= A_1 X_t + \dots + A_{q+1} X_{t-q} + \varepsilon_t \\ E \varepsilon_t \varepsilon_t' &= D \end{aligned} \tag{2}$$

$\varepsilon$  denotes the (k)-dimensional vector of structural innovations with the variance-covariance matrix  $D$ . The variables that are controlled by the policy maker are assumed to be exogenous, while the endogenous macroeconomic variables contained in  $Y$  are the ultimate goal of policy actions. Research was aimed at finding the optimal policy response to movements in macroeconomic variables to achieve specific targets for these macroeconomic variables.

Of course the feedback rule as specified in equation (1) can be easily estimated in the framework of such a multivariate structural model. However, this approach is subject to two prominent critiques. First, the Lucas-critique suggests that Cowles Commission models might be useless for policy simulation since they do not take expectations in account. We will discuss this critique in more detail in part II. Second, the Sims-critique says that exogeneity is attributed arbitrarily to some variables, while in a world of forward looking agents no variable can be treated as exogenous. This critique will be discussed in more detail in the next section. Furthermore there is no general agreement about the structure of such a model, which can also be seen as a reason for the popularity of VARs.

#### 2.1.2 Reduced Form Models: The St. Louis Model

On the other side, effects of monetary policy have been modelled in reduced form models, one of which is the widely known St. Louis model. The St. Louis model was developed in an article by Andersen and Jordan (1968), who were researchers at the Federal Reserve Bank of St. Louis. Using at that time state-of-the-art computerized regression programs, the authors tried to assess the relative importance of monetary and fiscal stimuli for real economic development. The resulting St. Louis equation is a reduced-form macroeconomic model linking contemporaneous and past changes in the monetary base ( $\Delta M$ ) and changes in "high-employment" fiscal expenditures ( $\Delta E$ ) to the change in nominal gross national product (GNP,  $\Delta Y$ ). "High-employment" fiscal expenditures of the U.S. Federal government were estimated expenditures, given some arbitrarily defined high-

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employment level of economic activity. Using quarterly US-data ranging from 1952 to 1968 the authors estimated the following relationship:

$$\begin{aligned} \Delta Y_t = & 2.28 + 1.54\Delta M_t + 1.56\Delta M_{t-1} + 1.44\Delta M_{t-2} + 1.29\Delta M_{t-3} \\ & + 0.40\Delta E_t + 0.54\Delta E_{t-1} - 0.03\Delta E_{t-2} - 0.74\Delta E_{t-3} \end{aligned} \quad (3)$$

The results of Andersen and Jordan (1968) led to controversial discussions, mainly for two reasons. First, the parameter estimates indicate that changes of the monetary base translate into changes in nominal GNP, while the effect of fiscal impulses is less pronounced. According to Andersen and Jordan (1968) the results indicate that the fiscal impulses are followed by a crowding out in the private sector, reflected in negative coefficients of the second and third lag of  $\Delta E$ . The cumulated effects of fiscal impulses are not significantly different from 0. This result was in contrast to the Keynesian consensus. Second, the authors used a reduced form model, which was in contrast to the prevailing large-scale structural models.

Romer (2006) notes two problems of the St. Louis equation. First, it is difficult to infer the direction of causality from observed correlation. E.g. if firms would plan to increase output and thus would demand liquid assets and loans, money could rise before output rises, but still we would not say that money causes output since it is rather the plans and expectations of firms that caused money. Second, failing to observe correlation does not necessarily mean that money does not cause output. E.g. the FED might adjust money to offset other factors that influence output. If this stabilization policy is successful, observable fluctuations in money would occur without subsequent fluctuations in output. Sims (1980) highlights an additional problem of the St. Louis type of model, namely that exogenous variables at the right side of the equation have to be truly exogenous, i.e. no feedback may occur between income and money and fiscal expenditures. This assumption is not reasonable. Thus, the true relation cannot be captured in a single equation. Having assessed traditional large-scale models and reduced form models of the St. Louis type, Sims (1980, 14) concludes: "Because existing large models contain too many incredible restrictions, empirical research aimed at testing competing macroeconomic theories too often proceeds in a single or few-equation framework. For this reason alone, it appears worthwhile to investigate the possibility of building large models in a style which does not tend to accumulate restrictions so haphazardly. (...) It should be feasible to estimate large-scale macromodels as unrestricted reduced forms, treating all variables as endogenous."

### 2.1.3 Contemporary Approaches

The logical consequence of the Sims critique is the development of VARs which are in essence multivariate models that model all variables endogenously. This approach will be introduced in section 3.

While we will discuss the identification of VARs in more detail later, it is clear that each statistical model imposes restrictions or identifying assumptions, such as the functional form of the monetary policy rule  $f(\cdot)$  or the choice of the lag length.

Note again that without a structural economic model it is the response of economic variables to exogenous monetary policy shocks  $\varepsilon_t^s$  that is of interest. This is because responses after endogenous policy actions (i.e. systematic policy actions based on the information set  $I_t$ ) may be caused by the policy actions themselves, but also by the economic developments that caused the monetary authority to act.

## 2.2 Narrative Approach

Source: Christiano et al. (1998), Romer and Romer (1989), Romer and Romer (1994)

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Statistical approaches may be considered to be relatively restrictive and the identified monetary policy shock may not be exogenous for various reasons. For example the functional form of estimated policy rule might be incorrect due to a misspecification of the policy maker's information set. Furthermore, the practical implementation requires various auxiliary assumptions such as the selection of lag structure or the order of differencing of variables. In addition, Romer and Romer (1989, 121) underline the difficulties of inferring the direction of causality, an issue that has been discussed in section 2.1.2.

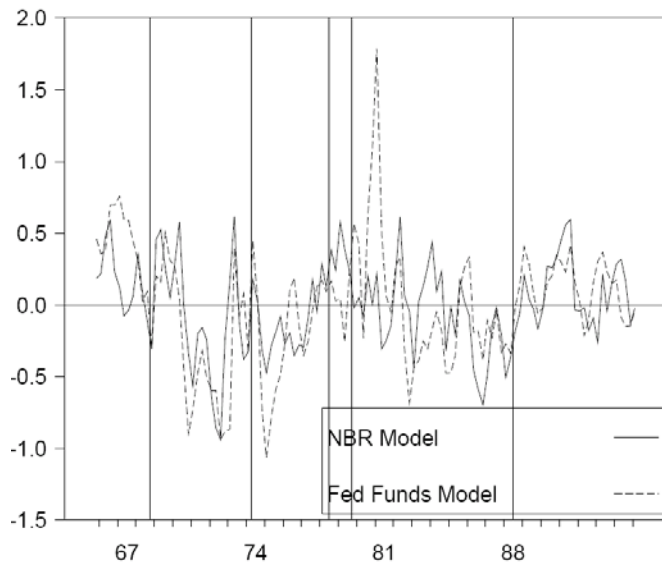
Consequently Romer and Romer (1989) suggest a "narrative approach" to identifying monetary policy shocks without statistical methods. In short, Romer and Romer (1989, 122) use "the historical record, such as the descriptions of the process and reasoning that led to decisions by the monetary authority and accounts of the sources of monetary disturbances, to identify episodes when there were large shifts in monetary policy or in the behaviour of the monetary sector that were not driven by developments on the real side of the economy. The test of whether monetary disturbances matter is then simply to see whether output is unusually low following negative shocks of this type (...)." More precisely, Romer and Romer (1989) only consider periods when the Federal Reserve specifically intended to use the tools it had available to attempt to create a recession to cure inflation as episodes of a monetary policy shock. A monetary policy shock is thus rather narrowly defined, since it only includes monetary shocks that are generated by concerns about inflation. One motive for such a narrow definition is that in these periods output should depend predominantly on monetary policy.

The definition implicitly entails two identifying assumptions regarding the relation of output and inflation. First, Romer and Romer (1989) assume that inflation does not exert a direct effect on output during the periods of contractionary monetary policy shocks. E.g. uncertainty, tax effects, or menu costs caused by inflation are not assumed to have direct effects on output. Second, the Federal Reserve is assumed to react directly to inflation and not to other shocks that might directly affect output.

Romer and Romer (1989) and Romer and Romer (1994) identify the following dates when the Federal reserve moved to induce a recession to reduce inflation: December 1968, April 1974, August 1978, October 1979, December 1988. Romer and Romer (1989, 135) find that "in each case the Federal Reserve appears to have made a deliberate decision to sacrifice real output to lower inflation".

Christiano et al. (1998) contrast the Romer and Romer (1989) dates with estimated monetary policy shocks. Figure 1 displays the series, where the vertical lines are the Romer and Romer (1994) dates. Christiano et al. (1998) use two models to estimate the monetary policy shock  $\varepsilon_t^m$ . In short, both models implement a policy rule as specified in equation (1) in a VAR framework, the policy instrument being the Federal Funds rate in the Fed Funds model, while non-borrowed reserves are the policy instrument in the NBR model. It is noteworthy that with the exception of 1979, each Romer and Romer episode is followed by a contractionary (i.e. positive) Fed Funds rate or non-borrowed reserves policy shock within one to two quarters.

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**Figure 1: Romer and Romer episodes (vertical lines) and monetary policy shocks estimated by feedback rules. Source: Christiano et al. (1998).**

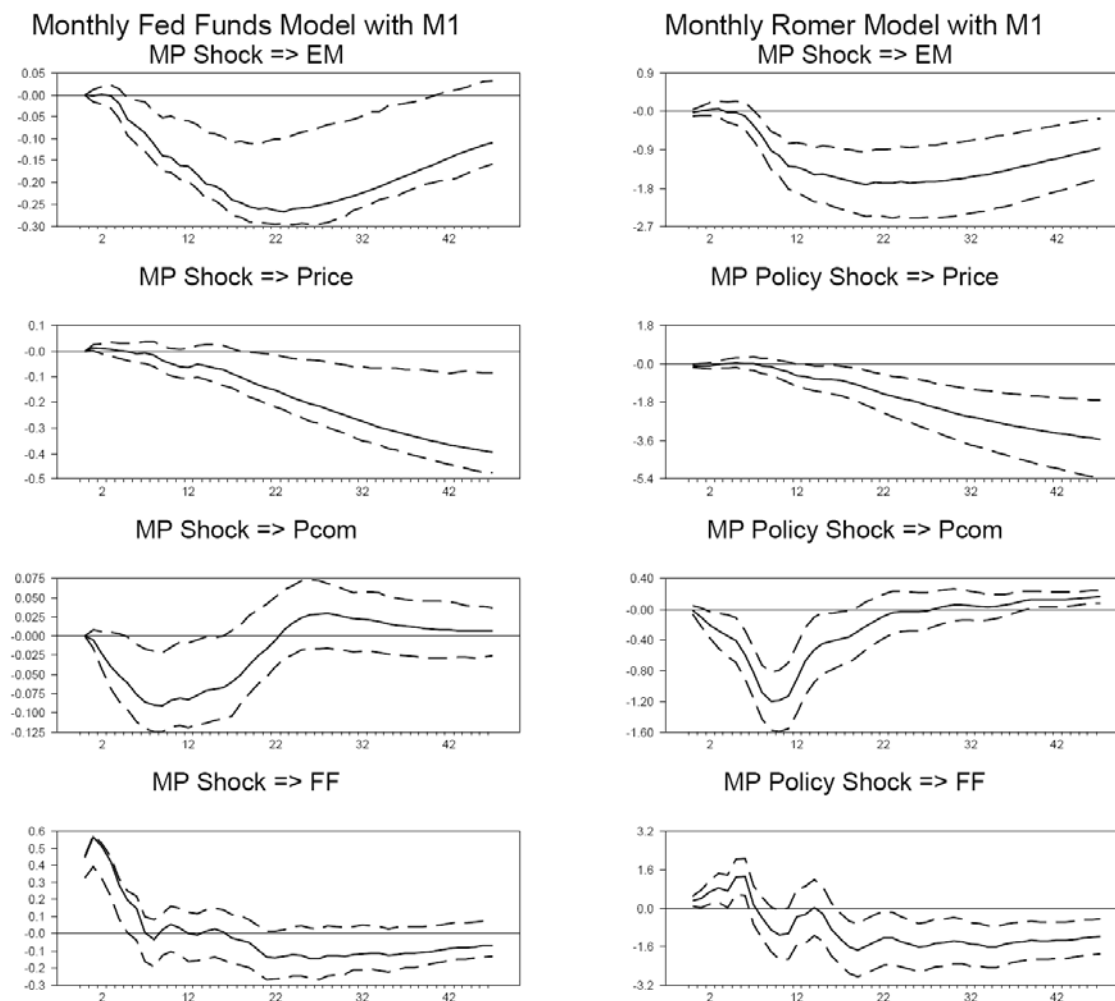
Christiano et al. (1998) capture the impact of a Romer and Romer (1998) shock on a set of economic variables  $Z_t$  using the following framework:

$$Z_t = B_0 + B_1 Z_{t-1} + \dots + B_q Z_{t-q} + C_0 d_t + \dots + C_p d_{t-p} + u_t \quad (4)$$

$$Z_t = \begin{pmatrix} z_{1,t} \\ \vdots \\ z_{k,t} \end{pmatrix}, B_0 = \begin{pmatrix} b_1^{(0)} \\ \vdots \\ b_k^{(0)} \end{pmatrix}, B_i = \begin{pmatrix} b_{1,1}^{(i)} & \dots & b_{1,k}^{(i)} \\ \vdots & \ddots & \vdots \\ b_{k,1}^{(i)} & \dots & b_{k,k}^{(i)} \end{pmatrix}, C_j = \begin{pmatrix} c_1^{(j)} \\ \vdots \\ c_k^{(j)} \end{pmatrix} \text{ for } i = 1 \dots q, j = 1 \dots p, u_t = \begin{pmatrix} u_{1,t} \\ \vdots \\ u_{k,t} \end{pmatrix}$$

where  $d$  is a dummy variable that is equal to 1 at a Romer and Romer (1998) date and 0 otherwise. Figure 2 displays the response of  $z_i$  to a Romer and Romer (1998) shock and contrasts it with the response to a monetary policy shock in the Fed Funds model framework. In both cases the responses to a contractionary monetary policy shock are very similar: the level of employment (EM) declines, the price level is only affected in the medium to long run (Pcom denotes commodity prices), and the short term interest rate (FF) rises.

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**Figure 2: The impact of a Romer and Romer (1989) shock. Source: Christiano et al. (1998).**

Obviously, the advantage of the narrative approach proposed by Romer and Romer (1989) is that no formal specification of a feedback rule is necessary. However, the two implicit identifying assumptions are relatively restrictive. Also, the approach only identifies few periods of monetary policy shocks and does not deliver any quantitative information on the extent of policy actions. Furthermore, recent evidence shows that the dummy variable generated by the narrative approach is predicted by macroeconomic variables other than the price level (Christiano et al. (1998, 61)). Thus the second identifying assumption might be violated.

### 2.3 Financial Markets Expectations

*Source: Favero (2001, 6.7.1), Rudebusch (1998)*

Recently various approaches to identify unanticipated monetary policy shocks using financial markets data have been proposed.

Rudebusch (1998) suggests calculating monetary policy shocks as the difference between the Fed Funds rate and the Fed Funds Futures rate which can be interpreted as the expected monthly average of the daily Fed Funds rate during the corresponding month. Using the sample 1988 to 1995, Rudebusch (1998) finds that the short term one month ahead Fed Funds Futures are efficiently priced:

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$$FFR_t = -0.045 + 1.00FFF1_{t-1}, \quad R^2 = 0.996, DW = 1.83$$

$$(0.0463) \quad (0.0074) \quad (5)$$

where  $FFR_t$  is the Fed Funds rate at month  $t$ , and  $FFF1_{t-1}$  is the rate of the month  $t$  Fed Funds Futures contract at month  $t - 1$ . The standard deviations in brackets indicate that there is no significant bias and that the slope is not significantly different from 1. Furthermore, the Durbin Watson statistic indicates that the residuals are uncorrelated. Rudebusch (1998) finds similar results for the 2 and 3 month ahead Fed Funds Futures contract. Consequently, Rudebusch (1998) constructs the one month and one quarter unanticipated monetary policy shock as follows:

$$u_t^{s,monthly} = FFR_t - FFF1_{t-1} \quad (6)$$

$$u_q^{s,quarterly} = (FFR_t + FFR_{t+1} + FFR_{t+2} - FFF1_{t-1} - FFF2_{t-1} - FFF3_{t-1})/3 \quad (7)$$

where the quarter  $q$  entails months  $t, t + 1, t + 2$ .  $FFF2$  and  $FFF3$  denote the 2 and 3 months ahead Fed Futures rate. Note that  $u^s$  comprises both exogenous and endogenous monetary policy shocks, since it simply measures surprises relative to information available at the end of month  $t - 1$ . Rudebusch (1998) tries to identify exogenous monetary policy shocks  $\varepsilon_t^s$  by regressing  $u_q^s$  on a constant and the surprise component of employment numbers  $E_t$  (defined as the difference between published employment numbers at month  $t$  and expected employment numbers at  $t - 1$ ). This yields:

$$u_t^{s,monthly} = -0.040 + 0.00028E_t + \varepsilon_t^{s,monthly}, \quad R^2 = 0.043$$

$$(0.017) \quad (0.00015) \quad (8)$$

The  $t$ -values indicate that regression only has limited explanatory power, but Rudebusch (1998) still considers it as a first step in orthogonalizing  $u_t^{s,monthly}$ .

Figure 3 displays the estimated monetary policy shocks together with Fed Funds shocks estimated in a reduced form VAR, while Figure 4 displays the exogenous, structural monetary policy shocks. We see that the VAR-shocks are more volatile than the Futures rates shocks. Also, there is no apparent comovement between the two series. Rudebusch (1998) concludes that the estimated VAR-shocks are not significantly related to financial markets' perception of monetary policy shocks.

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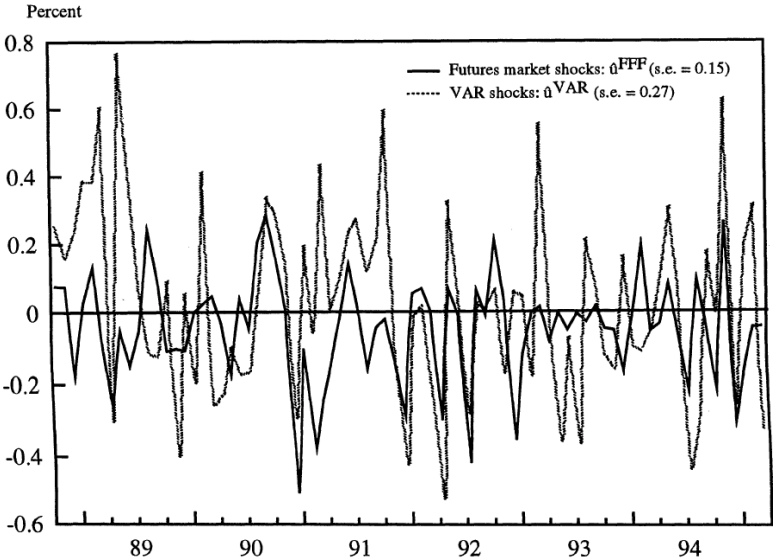


Figure 3: Monthly Futures Markets and VAR Unanticipated Shocks. Source: Rudebusch (1998).

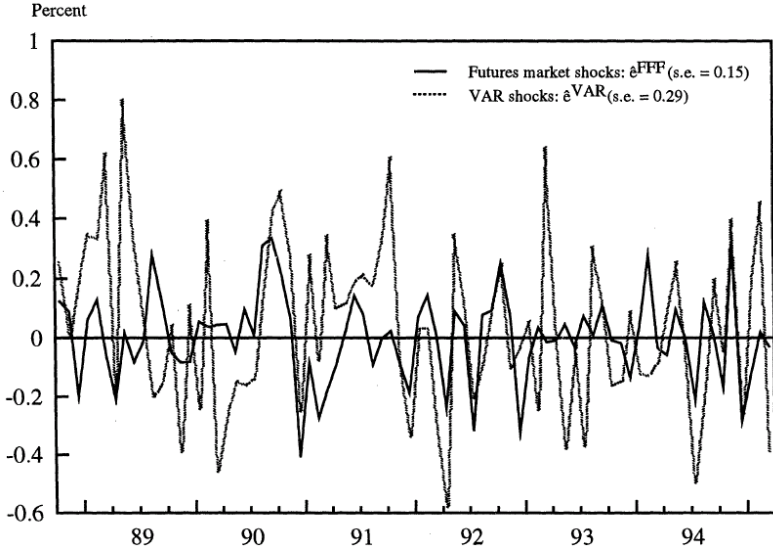


Figure 4: Monthly Futures Markets and VAR Exogenous (Structural) Shocks. Source: Rudebusch (1998).

Favero (2001, 192) discusses two additional approaches. One approach measures unanticipated monetary policy shocks as the change of the target rate on the day of policy announcement. Figure 5 illustrates the effects of the Swiss National Bank's monetary policy decision of September 15<sup>th</sup>, 2005 to leave the target rate at 0.75%. Of course, this approach is relatively fragile due to confounding effects of other news that are released at the same time. Another approach bases on the relation between spot rates and instantaneous forward rates, see Favero (2001, 192) for details and further references.

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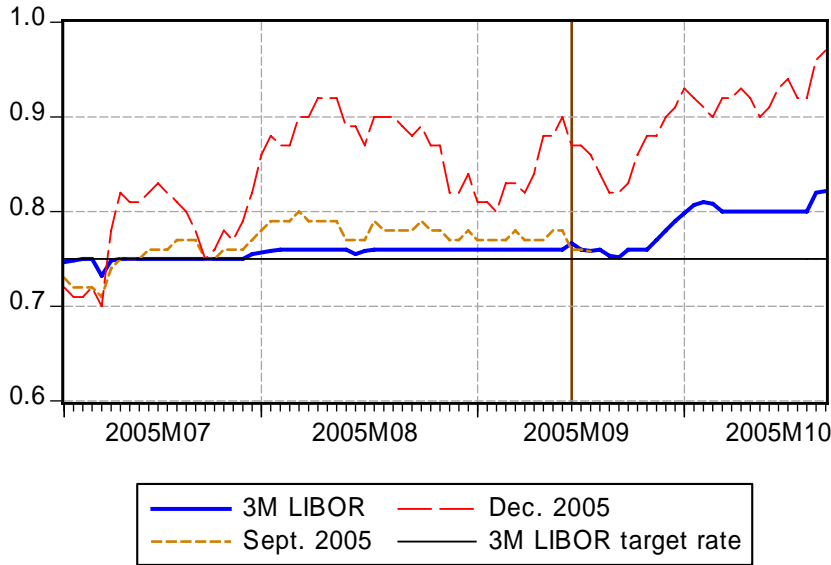


Figure 5: LIBOR, Futures, and SNB Target Rate.

## 3 The VAR Approach

### 3.1 Introduction

Source: Hamilton (1994, 10.1, 11.4), Enders (2004, 5.5-7)

#### 3.1.1 Reduced Form Representation of a VAR

A vector autoregression (VAR) is a system of equations in which each variable is regressed on a constant and on  $q$  lags of itself as well as  $q$  lags of all other variables. In contrast to traditional multiequation models, all variables are endogenous. In its reduced form (also called standard form) a VAR of order  $q$  is given by:

$$Z_t = B_0 + B_1 Z_{t-1} + B_2 Z_{t-2} + \dots + B_q Z_{t-q} + u_t \quad (9)$$

$$Z_t = \begin{pmatrix} z_{1,t} \\ \vdots \\ z_{k,t} \end{pmatrix}, B_0 = \begin{pmatrix} b_1^{(0)} \\ \vdots \\ b_k^{(0)} \end{pmatrix}, B_i = \begin{pmatrix} b_{1,1}^{(i)} & \dots & b_{1,k}^{(i)} \\ \vdots & \ddots & \vdots \\ b_{k,1}^{(i)} & \dots & b_{k,k}^{(i)} \end{pmatrix} \text{ for } i = 1 \dots q, u_t = \begin{pmatrix} u_{1,t} \\ \vdots \\ u_{k,t} \end{pmatrix}$$

Where  $Z$  is a  $(k)$ -dimensional vector of endogenous variables,  $B_0$  is a  $(k)$ -dimensional vector of constants, and  $B_1 \dots B_q$  are  $(k \times k)$  dimensional autoregressive coefficient matrices.  $u_t$  is a  $(k)$ -dimensional vector of normally distributed error terms with the following properties:

$$\begin{aligned} E u_t &= 0 \\ E u_t u_t' &= V \\ E u_t u_s' &= 0, \text{ if } t \neq s \end{aligned} \quad (10)$$

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$V$  denotes the variance-covariance matrix of  $u_t$ . As each variable depends on predetermined lagged terms and as the error terms are assumed to be serially uncorrelated with constant variance, the reduced form VAR can be estimated using OLS.

### 3.1.2 Moving Average Representation

The VAR( $q$ ) can be rewritten as an VAR(1). Note, that a stationary multiple time series can be written in terms of deviation from its expected value  $\mu$ . Taking the expectations operator on both sides of our VAR( $q$ ) gives:

$$\begin{aligned} E(Z_t) &= E(B_0 + B_1 Z_{t-1} + B_2 Z_{t-2} + \dots + B_q Z_{t-q} + u_t) \\ E(Z_t) &= \mu = B_0 + B_1 \mu + B_2 \mu + \dots + B_q \mu \end{aligned} \quad (11)$$

Hence:

$$(Z_t - \mu) = B_1(Z_{t-1} - \mu) + B_2(Z_{t-2} - \mu) + \dots + B_q(Z_{t-q} - \mu) + u_t \quad (12)$$

We can rewrite the above VAR( $q$ ) as VAR(1):

$$\tilde{Z}_t = \tilde{B}\tilde{Z}_{t-1} + \tilde{u}_t \quad (13)$$

$$\tilde{Z}_t = \begin{pmatrix} Z_t - \mu \\ Z_{t-1} - \mu \\ Z_{t-2} - \mu \\ \vdots \\ Z_{t-q+1} - \mu \end{pmatrix}, \tilde{B} = \begin{pmatrix} B_1 & B_2 & \dots & B_{q-1} & B_q \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{pmatrix}, \tilde{u}_t = \begin{pmatrix} u_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Repeated substitution results in:

$$\tilde{Z}_{t+s} = \tilde{B}^s \tilde{Z}_t + \tilde{u}_{t+s} + \tilde{B}\tilde{u}_{t+s-1} + \dots + \tilde{B}^{s-1}\tilde{u}_t \quad (14)$$

The VAR is stationary, if all eigenvalues of  $\tilde{B}$  lie inside the unit circle.<sup>1</sup> Then it holds that  $\tilde{B}^s \rightarrow 0$  as  $s \rightarrow \infty$ . Thus we can write  $Z_{t+s}$  as the convergent sum of the history of  $u_t$ . Following,  $\tilde{Z}_t^{[1\dots k, 1\dots k]}$  is a vector of the first  $k$  rows of  $\tilde{Z}_t$ ,  $\Psi_j$  denotes the upper left block of  $\tilde{B}^j$  consisting of the first  $k$  rows and  $k$  columns of  $\tilde{B}^j$ :

$$\begin{aligned} Z_t &= \mu + \tilde{Z}_t^{[1\dots k, 1\dots k]} \\ &= \mu + u_t + \Psi_1 u_{t-1} + \Psi_2 u_{t-2} + \Psi_3 u_{t-3} + \dots \end{aligned} \quad (15)$$

This is the vector  $MA(\infty)$  representation of the VAR( $q$ ). Alternatively, the moving average representation can be obtained using the lag operator. Applying the lag operator, the reduced form VAR of equation (9) can be written as follows:

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<sup>1</sup> See Hamilton (1994, 259).

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$$(I - B_1L - B_2L^2 - \dots - B_qL^q)Z_t = B_0 + u_t \quad (16)$$

where  $L^j Z_t = Z_{t-j}$ . The above equation can be written more compactly:

$$B(L)Z_t = B_0 + u_t \quad (17)$$

where  $B(L)$  is a  $(k \times k)$ -matrix of polynomials in the lag operator  $L$ . The row  $i$ , column  $j$  element of  $B(L)$  is given by:

$$B(L)_{i,j} = \delta_{ij} + b_{i,j}^{(1)} + b_{i,j}^{(2)} + \dots + b_{i,j}^{(q)}, \text{ where } \delta_{ij} = 1 \text{ if } i = j, \text{ else } \delta_{ij} = 0 \quad (18)$$

If all eigenvalues of  $B(L)$  lie inside the unit circle, the VAR(q) has an  $MA(\infty)$  representation which is given by:

$$\begin{aligned} Z_t &= B(L)^{-1}B_0 + B(L)^{-1}u_t \\ &= \mu + \Psi(L)u_t \end{aligned} \quad (19)$$

The moving-average coefficients can be calculated using the method of undetermined coefficients, based on the following relationship: <sup>2</sup>

$$\begin{aligned} \Psi(L) &= B(L)^{-1} \\ B(L)\Psi(L) &= I \\ (I - B_1L - B_2L^2 - \dots - B_qL^q)(I + \Psi_1L + \Psi_2L^2 + \dots) &= I \end{aligned} \quad (20)$$

### 3.1.3 Impulse-Response Function

The vector moving average representation of the VAR(q) expresses the endogenous variable  $Z_t$  as a linear function of current and past innovations  $u$ :

$$Z_t = \mu + u_t + \Psi_1u_{t-1} + \Psi_2u_{t-2} + \Psi_3u_{t-3} + \dots \quad (15)$$

The matrix  $\Psi_s$  has the interpretation

$$\frac{\partial Z_{t+s}}{\partial u_t} = \Psi_s \quad (21)$$

We define  $\psi_{i,j}(s)$  as the row  $i$ , column  $j$  element of  $\Psi_s$ . The coefficient  $\psi_{i,j}(s)$  is called impact multiplier. It quantifies the consequence of a one unit increase in the error term  $u_{j,t}$  of variable  $z_j$  for the value of  $z_{i,t+s}$ , all other innovations held constant.  $\psi_{i,j}(s)$  as function of  $s$  is called impulse response function. Plotting  $\psi_{i,j}(s)$  against  $s$  visualizes the behaviour of  $z_i$  in response to a shock in  $u_{j,t}$ . The combined effect of changes in  $u_{1,t}$  through  $u_{k,t}$  on the vector  $Z$  is given by:

<sup>2</sup> See Hamilton (1994, 260) for more details on the method of undetermined coefficients.

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$$\Delta Z_{t+s} = \Psi_s \Delta u_t \quad (22)$$

The sum

$$\sum_{s=0}^S \psi_{i,j}(s) \quad (23)$$

reflects the effect of a one-unit shock in  $u_{j,t}$  on  $z_i$  up to time  $t+S$ . As a function of  $S$  this sum is called the cumulative impulse response function. Note that for  $S \rightarrow \infty$  the sum is finite since  $z_i$  is assumed to be stationary.

### 3.2 VAR as a Reduced Form Structural Model

Source: Hamilton (1994, 11.6)

A traditional structural model relates the  $(k)$ -dimensional vector  $Y$  of endogenous variables to contemporaneous and lagged realisations of exogenous variables of the  $(j)$ -dimensional vector  $X$  :

$$\begin{aligned} A_0 Y_t &= A_1 X_t + \dots + A_{q+1} X_{t-q} + \varepsilon_t \\ E \varepsilon_t \varepsilon_t' &= D \end{aligned} \quad (2)$$

Where  $\varepsilon$  ist the  $(k)$ -dimensional vector structural innovations, and  $D$  is the variance-covariance matrix of the structural innovations. As an example we consider a simplified New Keynesian simultaneous dynamic equation model that captures output-gap  $y$ , inflation  $\pi$ , and interest rate  $i$  :

$$\begin{aligned} y_t &= \alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - E_{t-1}\pi_t) + \varepsilon_{1,t} \\ \pi_t &= \pi_{t-1} + \gamma y_t + \varepsilon_{2,t} \\ i_t &= \lambda y_t + \varphi \pi_t + \varepsilon_{3,t} \end{aligned} \quad (24)$$

The first equation is an IS curve and the second equation can be interpreted as a Phillips-curve relationship. The third equation defines monetary policy reaction function (Taylor rule), where  $\varepsilon_t$  is a monetary policy shock. The model assumes rational expectations, which however only appear in the output-equation. Solving for expected inflation yields:

$$\begin{aligned} E_{t-1}\pi_t &= \pi_{t-1} + \gamma E_{t-1}y_t + E_{t-1}\varepsilon_{2,t} \\ E_{t-1}y_t &= \alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - E_{t-1}\pi_t) + E_{t-1}\varepsilon_{1,t} \\ &= \alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - \pi_{t-1} - \gamma E_{t-1}y_t) \\ &= \frac{\alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - \pi_{t-1})}{1 - \vartheta\gamma} \end{aligned}$$

Substituting the result into the output-equation yields:

## Identifying monetary policy shocks

$$\begin{aligned}
 y_t &= \alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - \pi_{t-1} - \gamma \frac{\alpha y_{t-1} + \beta y_{t-2} - \vartheta(i_{t-1} - \pi_{t-1})}{1 - \vartheta\gamma}) + \varepsilon_{1,t} \\
 &= \frac{\alpha}{1 - \vartheta\gamma} y_{t-1} + \frac{\beta}{1 - \vartheta\gamma} y_{t-2} - \frac{\vartheta}{1 - \vartheta\gamma} i_{t-1} + \frac{\vartheta}{1 - \vartheta\gamma} \pi_{t-1} + \varepsilon_{1,t}
 \end{aligned}$$

The model can be rewritten in matrix notation:

$$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ -\lambda & -\varphi & 1 \end{pmatrix} \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{1 - \theta\gamma} & \frac{\theta}{1 - \theta\gamma} & -\frac{\theta}{1 - \theta\gamma} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \begin{pmatrix} \frac{\beta}{1 - \theta\gamma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

The reduced form which can be estimated by OLS is thus given by:

$$\begin{aligned}
 \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ -\lambda & -\varphi & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\alpha}{1 - \theta\gamma} & \frac{\theta}{1 - \theta\gamma} & -\frac{\theta}{1 - \theta\gamma} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \\
 &+ \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ -\lambda & -\varphi & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\beta}{1 - \theta\gamma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ -\lambda & -\varphi & 1 \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix} \\
 \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} &= \begin{pmatrix} \frac{\alpha}{-\theta\gamma + 1} & \frac{\theta}{-\theta\gamma + 1} & -\frac{\theta}{-\theta\gamma + 1} \\ \alpha \frac{\gamma}{-\theta\gamma + 1} & \theta \frac{\gamma}{-\theta\gamma + 1} + 1 & -\theta \frac{\gamma}{-\theta\gamma + 1} \\ \alpha \frac{\lambda + \varphi\gamma}{-\theta\gamma + 1} & \varphi + \theta \frac{\lambda + \varphi\gamma}{-\theta\gamma + 1} & -\theta \frac{\lambda + \varphi\gamma}{-\theta\gamma + 1} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \\
 &+ \begin{pmatrix} \frac{\beta}{-\theta\gamma + 1} & 0 & 0 \\ \beta \frac{\gamma}{-\theta\gamma + 1} & 0 & 0 \\ \beta \frac{\lambda + \varphi\gamma}{-\theta\gamma + 1} & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ (\lambda + \varphi\gamma) & \varphi & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix} \tag{25}
 \end{aligned}$$

As a second example we consider a simultaneous dynamic equation model that captures money supply  $m$ , output  $y$ , and interest rate  $i$ :

$$\begin{aligned}
 m_t &= a_{1,2}^{(0)} y_t + a_{1,3}^{(0)} i_t + a_{1,2}^{(1)} m_{t-1} + \dots + a_{1,2}^{(q)} m_{t-q} + a_{1,1}^{(1)} y_{t-1} + \dots + a_{1,1}^{(q)} y_{t-q} + a_{1,3}^{(1)} i_{t-1} + \dots + a_{1,3}^{(q)} i_{t-q} + \varepsilon_{1,t} \\
 y_t &= a_{2,1}^{(0)} m_t + a_{2,3}^{(0)} i_t + a_{2,2}^{(1)} m_{t-1} + \dots + a_{2,2}^{(q)} m_{t-q} + a_{2,2}^{(1)} y_{t-1} + \dots + a_{2,2}^{(q)} y_{t-q} + a_{2,3}^{(1)} i_{t-1} + \dots + a_{2,3}^{(q)} i_{t-q} + \varepsilon_{2,t} \\
 i_t &= a_{3,1}^{(0)} m_t + a_{3,2}^{(0)} y_t + a_{3,1}^{(1)} m_{t-1} + \dots + a_{3,1}^{(q)} m_{t-q} + a_{3,2}^{(1)} y_{t-1} + \dots + a_{3,2}^{(q)} y_{t-q} + a_{3,3}^{(1)} i_{t-1} + \dots + a_{3,3}^{(q)} i_{t-q} + \varepsilon_{3,t}
 \end{aligned} \tag{26}$$

The first equation can be interpreted as a money demand function. We assume that money demand depends on current income and interest rates, as well as past realizations of all three variables. However, it is not possible to

## Identifying monetary policy shocks

estimate this equation by OLS due to a simultaneous equation bias.<sup>3</sup> OLS would yield inconsistent estimates, as it would simply summarize the correlations between the variables. However, the reaction of money demand to income and interest rates is only one reason for the observed correlation. Another source of correlation results because the interest rate as well as output is likely to react to money supply. For example, unusually large money demand shock  $\varepsilon_{1,t}$  causes  $m_t$  to rise. At the same time however, the high money demand will likely affect the interest rate  $i_t$ . Thus,  $\varepsilon_{1,t}$  is correlated with the regressor  $i_t$  and the first equation cannot be estimated by OLS. Hence, we should consider all variables at time  $t$  as endogenous and we need to introduce two more equations. The second equation relates the level of output/income to the money supply and interest rates. The third equation relates the interest rate to the level of money supply and income. The disturbance terms capture factors that are not considered by the model. Note that the third equation could also be interpreted as the reaction function of a central bank that sets an interest rate target with regard to current and past money supply, output, and interest rates. In this case  $\varepsilon_{3,t}$  could be interpreted as a money supply shock.

Both equation systems can be rewritten in matrix-form:

$$A_0 Z_t = C_0 + A_1 Z_{t-1} + A_2 Z_{t-2} + \dots + A_q Z_{t-q} + \varepsilon_t \quad (27)$$

$$Z_t = \begin{pmatrix} z_{1,t} \\ \vdots \\ z_{k,t} \end{pmatrix}, C_0 = \begin{pmatrix} c_1^{(0)} \\ \vdots \\ c_k^{(0)} \end{pmatrix}, A_i = \begin{pmatrix} a_{1,1}^{(i)} & \dots & a_{1,k}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{k,1}^{(i)} & \dots & a_{k,k}^{(i)} \end{pmatrix} \text{ for } i = 0 \dots q, \varepsilon_t = \begin{pmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{k,t} \end{pmatrix}$$

where  $Z$  is a  $(k)$ -dimensional vector of variables,  $A_0$  is a  $(k \times k)$  dimensional matrix with contemporaneous coefficients,  $C_0$  is a  $(k)$ -dimensional vector of constants, and  $A_1 \dots A_q$  are  $(k \times k)$  dimensional autoregressive coefficient matrices.  $u_t$  is a  $(k)$ -dimensional vector of normally distributed error terms.

If we premultiply each side by  $A_0^{-1}$  we get:

$$\begin{aligned} Z_t &= A_0^{-1} C_0 + A_0^{-1} A_1 Z_{t-1} + A_0^{-1} A_2 Z_{t-2} + \dots + A_0^{-1} A_q Z_{t-q} + A_0^{-1} \varepsilon_t \\ Z_t &= B_0 + B_1 Z_{t-1} + B_2 Z_{t-2} + \dots + B_q Z_{t-q} + u_t \end{aligned} \quad (28)$$

Equation (28) is the vector autoregressive representation of the structural dynamic equation system. Hence, a VAR can be seen as a reduced form of a general structural model.

### 3.3 Structural Form VAR

Source: Hamilton (1994, 11.6), Enders (2004, 5.10)

A structural form SVAR( $q$ ) in its most general form is given by:

$$\begin{aligned} A_0 Z_t &= C + A_1 Z_{t-1} + A_2 Z_{t-2} + \dots + A_q Z_{t-q} + S \varepsilon_t \\ E \varepsilon_t \varepsilon_t' &= I \end{aligned} \quad (29)$$

<sup>3</sup> See Hamilton (1994, 326) for more details on the simultaneous equation bias in this context.

## Identifying monetary policy shocks

where the previously introduced notation applies.  $\varepsilon_t$  is a vector of pairwise uncorrelated structural innovations with unit variance.  $E\varepsilon_t\varepsilon_t'$  is the variance-covariance matrix of structural innovations which is equal to the identity matrix.  $S$  is a  $(k \times k)$  matrix that specifies which variables are to what extent directly affected by structural shocks. Note that  $S$  typically is a diagonal matrix. In the context of SVARs, structural innovations are considered to be exogenous forces that are not directly observed by the econometrician. This is in contrast to the dynamic simultaneous equation approach, where structural innovations are interpreted as error terms, reflecting the influences of factors not considered by the model.

The coefficients of the structural and reduced form are related as follows:

$$\begin{aligned} B_i &= A_0^{-1}A_i, i = 1 \dots q \\ B_0 &= A_0^{-1}C \\ A_0^{-1}S\varepsilon_t &= u_t \\ V &= A_0^{-1}SIS'(A_0^{-1})' \end{aligned} \tag{30}$$

Respectively:

$$\begin{aligned} A_i &= A_0B_i, i = 1 \dots q \\ C &= A_0B_0 \\ S^{-1}A_0u_t &= \varepsilon_t \end{aligned} \tag{31}$$

If all eigenvalues of  $B(L)$  lie inside the unit circle, the  $MA(\infty)$  representation is given by:

$$\begin{aligned} Z_t &= B(L)^{-1}B_0 + B(L)^{-1}A_0^{-1}SS^{-1}A_0u_t \\ &= \mu + \Phi(L)\varepsilon_t \end{aligned} \tag{32}$$

Consequently, the impulse response functions can be derived from the following relationship:

$$\frac{\partial Z_{t+s}}{\partial \varepsilon_t} = \Psi_s A_0^{-1} S = \Phi_s \tag{33}$$

We define  $\varphi_{i,j}(s)$  as the row  $i$ , column  $j$  element of  $\Phi_s$ . The coefficient  $\varphi_{i,j}(s)$  is called impact multiplier. It quantifies the consequence of a one unit increase in the error term  $\varepsilon_{j,t}$  of variable  $z_j$  for the value of  $z_{i,t+s}$ , all other innovations held constant.  $\varphi_{i,j}(s)$  as function of  $s$  is called impulse response function. The sum

$$\sum_{s=0}^S \varphi_{i,j}(s) \tag{34}$$

reflects the effect of a one-unit shock in  $\varepsilon_{j,t}$  on  $z_i$  up to time  $t+S$ . As a function of  $S$  this sum is called the cumulative impulse response function. For  $S \rightarrow \infty$  the sum is finite since  $z_i$  is assumed to be stationary.

### 3.4 The Identification Problem

## Identifying monetary policy shocks

It can be easily shown that an infinite set of different values of  $A_0$  and  $A_1$  through  $A_q$  results in observationally equivalent reduced form models. To see this, we premultiply the structural form with a full rank ( $k \times k$ ) matrix  $Q$  :

$$QA_0Z_t = QC + QA_1Z_{t-1} + \dots + QA_{q+1}Z_{t-q} + QS\varepsilon_t \quad (35)$$

Transformation of the model into its reduced form yields:

$$\begin{aligned} Z_t &= A_0^{-1}Q^{-1}QC + \dots + A_0^{-1}Q^{-1}QA_{q+1}Z_{t-q} + A_0^{-1}Q^{-1}QS\varepsilon_t \\ EA_0^{-1}Q^{-1}QS\varepsilon_t(A_0^{-1}Q^{-1}QS\varepsilon_t)' &= A_0^{-1}SIS'(A_0^{-1})' \end{aligned} \quad (36)$$

$Q$  cancels out, which implies that the above model is not identified. Without imposing additional restrictions, the structural parameters cannot be inferred from the estimated reduced form coefficients, since there is more than one structural model that leads to the same statistical model. Hence, the structural model is not identifiable. Identifiability would require that the  $Q$  matrix is identity. As the matrix  $Q$  has  $k \times k$  elements, we need to impose  $k \times k$  restrictions in order to identify the model.

The SVAR specification in equation (29) contains  $k$  constants in  $C$ ,  $k \times k \times q$  AR-coefficients in  $A_1 \dots A_q$ ,  $k \times k$  coefficients of contemporaneous relations in  $A_0$ , and  $k \times k$  elements of in  $S$ . The estimation of (9) yields  $k$  constants in  $B_0$ ,  $k \times k \times q$  AR-coefficients in  $B_1 \dots B_q$ , and  $(k+1)k/2$  elements of the estimated variance-covariance matrix  $V$ . Thus we obtain  $k + k \times k \times q + (k+1)k/2$  parameters to identify  $k + k \times k \times q + k \times k \times k$  structural coefficients. If  $S$  is restricted to be a diagonal matrix, then we need to impose an additional  $(k+1)k/2$  restrictions, of which  $k$  restrictions can be considered as normalization restrictions (usually  $\text{diag}(S)=1$  or  $\text{diag}(A_0)=1$ ).

## 4 Identification of a SVAR

### 4.1 Short Run Restrictions

#### 4.1.1 Choleski Factorization

*Source: Hamilton (1994, 91), Favero (2001, 6.2), Sims (1980, 21)*

Sims (1980) suggested an identification strategy that bases on the Choleski decomposition of the variance-covariance matrix  $V$ . The Choleski decomposition is a triangular factorization that applies to any symmetric, positive definite ( $k \times k$ ) matrix  $V$ :

$$V = A_0^{-1}SS'A_0^{-1'} = A_0^{-1}S(A_0^{-1}S)' = PP' \quad (37)$$

## Identifying monetary policy shocks

$$S = \begin{pmatrix} s_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & s_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & s_{k,k} \end{pmatrix}, A_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1}^{(0)} & 1 & 0 & \cdots & 0 \\ a_{3,1}^{(0)} & a_{3,2}^{(0)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{k,1}^{(0)} & a_{k,2}^{(0)} & \cdots & a_{k,k-1}^{(0)} & 1 \end{pmatrix}^{-1} \quad (38)$$

Where  $A_0^{-1}$  is a lower triangular matrix with diagonal elements equal to 1.  $S$  is a diagonal matrix with strictly positive diagonal elements.

The Choleski decomposition imposes  $k$  normalization restrictions (diagonal elements of  $A_0$  are equal to 1) and restricts an additional  $k(k-1)/2$  elements of  $A_0$  to 0. Thus it imposes a total of  $k(k+1)/2$  restrictions on the system, which just identifies the structural form. Obviously, the lower triangular matrix  $A_0$  implies a recursive structure: while  $z_{1,t}$  only depends on lagged realizations of  $Z$ ,  $z_{k,t}$  is the most endogenous variable and as such depends contemporaneously on  $z_{1,t} \dots z_{k-1,t}$ . This is reflected in the composition of reduced-form error terms:

$$A_0^{-1}S\varepsilon_t = u_t = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1}^{(0)} & 1 & 0 & \cdots & 0 \\ a_{3,1}^{(0)} & a_{3,2}^{(0)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{k,1}^{(0)} & a_{k,2}^{(0)} & \cdots & a_{k,k-1}^{(0)} & 1 \end{pmatrix}^{-1} \begin{pmatrix} s_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & s_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & s_{k,k} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \\ \vdots \\ \varepsilon_{k,t} \end{pmatrix} \quad (39)$$

### 4.1.2 Structural Restrictions

Source: Favero (2001, 6.6.2)

While the Choleski-decomposition does not need any a priori information (apart from the ordering of the variables),  $A_0$  can be restricted based on theoretical assumptions about the economic structure among the endogenous variables. An example is given in section 4.3 where we present a simplified version of the SVAR currently used at the Swiss National Bank.

Note at this point that if the aim of a SVAR is to evaluate competing macroeconomic models, the identifying restrictions should be independent of theoretical predictions of these models.

## 4.2 Long Run Restrictions

Source: Favero (2001, 6.2.3)

Long-run restrictions are restrictions on the cumulative impulse response function. Above we have derived the  $MA(\infty)$  representation of a SVAR, where the impulse response function bases on the following relationship:

$$\frac{\partial Z_{t+s}}{\partial \varepsilon_t} = \Psi_s A_0^{-1} S = \Phi_s \quad (40)$$

## Identifying monetary policy shocks

As function of  $s$  the row  $i$ , column  $j$  element of  $\Phi_s$ ,  $\varphi_{i,j}(s)$ , is called impulse response function. The cumulative impulse response function is given by:

$$\sum_{s=0}^S \varphi_{i,j}(s) \quad (41)$$

For  $S \rightarrow \infty$  the sum is finite since  $z_i$  is assumed to be stationary. In terms of the lag-polynomial  $\Phi(L) = 1 + \Phi_1 L + \Phi_2 L^2 + \dots$  we can write:

$$\sum_{s=0}^{\infty} \varphi_{i,j}(s) = \Phi(1)_{i,j} \quad (42)$$

Where  $\Phi(1)_{i,j}$  denotes the row  $i$ , column  $j$  element of  $\Phi(1) = 1 + \Phi_1 + \Phi_2 + \dots$ . E.g. if  $\Phi(1)_{i,j}$  is restricted to be 0, then the overall impact of shock  $j$  on variable  $i$  is zero.

We can implement long run restrictions given our short run restrictions in  $A_0$  by choosing  $S$  accordingly, since  $\Psi(1)A_0^{-1}S = \Phi(1)$ .

### 4.3 An Example: A Simplified Version of the Swiss National Bank's SVAR

Source: Jordan et al. (2002)

The model has the following structure:

$$\begin{pmatrix} a_{1,1}^{(0)} & a_{1,2}^{(0)} & 0 & 0 \\ a_{2,1}^{(0)} & a_{2,2}^{(0)} & 0 & 0 \\ a_{3,1}^{(0)} & a_{3,2}^{(0)} & a_{3,3}^{(0)} & a_{3,4}^{(0)} \\ a_{4,1}^{(0)} & a_{4,2}^{(0)} & a_{4,3}^{(0)} & a_{4,4}^{(0)} \end{pmatrix} \begin{pmatrix} \Delta \log p_t \\ \Delta \log y_t \\ \Delta \log m_t \\ \Delta r_t \end{pmatrix} = C + A_1 Z_{t-1} + \dots + A_q Z_{t-q} + S \varepsilon_t \quad (43)$$

$$\Phi(1) = \begin{pmatrix} \Phi(1)_{1,1} & \Phi(1)_{1,2} & \Phi(1)_{1,3} & \Phi(1)_{1,4} \\ 0 & \Phi(1)_{2,2} & 0 & 0 \\ \Phi(1)_{3,1} & \Phi(1)_{3,2} & \Phi(1)_{3,3} & \Phi(1)_{3,4} \\ 0 & \Phi(1)_{4,2} & 0 & \Phi(1)_{4,4} \end{pmatrix} \quad (44)$$

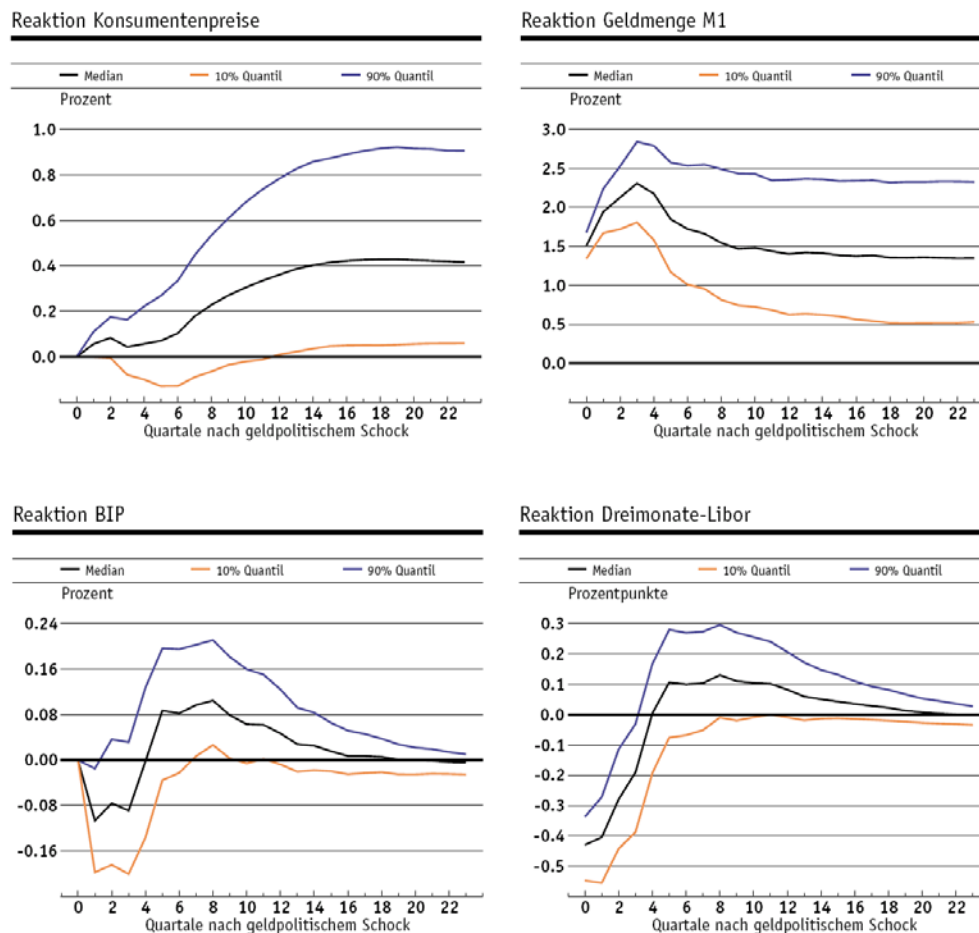
where  $p_t$  is the CPI,  $y_t$  is the real GDP,  $m_t$  is the monetary aggregate M1, and  $r_t$  denotes the 3 month LIBOR. Since all variables are non-stationary, the model is estimated in first differences.

While the inflation rate is the ultimate goal of monetary policy, economic development as captured by GDP is considered as well. Short term rates are the operating goal of monetary policy. M1 was added since this monetary aggregate responds quickly to monetary policy operations. Since M1 directly captures monetary policy actions,  $\varepsilon_{3,t}$  is considered to be the monetary policy shock. Note that the exchange rate is not considered

## Identifying monetary policy shocks

In the short run it is assumed that price level  $p_t$  and output  $y_t$  respond only with a lag of one quarter to money and interest rates. Monetary policy shocks however have contemporaneous effects on the monetary aggregate  $m_t$ , whereas money demand shocks also may affect the monetary policy instrument  $r_t$ . In the long run it is assumed that the real GDP is only affected by its own shocks, while the short rate is affected by interest rates shocks and GDP shocks.

The resulting impulse response functions are displayed in Figure 6.



**Figure 6: Impulse response functions of the SNB SVAR model to a monetary policy shock, estimation period Q2:1974 – Q2:2002.**

The structural VAR model can be used to calculate conditional forecasts. The official inflation forecast of the Swiss National Bank is such a conditional forecast in that it assumes a fixed short term rate  $r_t$  (i.e. a fixed 3M LIBOR). The procedure to calculate a conditional forecast is relatively simple. First, the unconditional forecast of the SVAR model for  $t+1$  is calculated ( $E_t r_{t+1}$ ). Second, the monetary policy shock in period  $t+1$  is adjusted so that – given all other shocks - the desired short term rate  $r^*$  results:

$$\varepsilon_{3,t+1} = \frac{(r^* - E_t r_{t+1})}{\varphi_{4,3}(1)} \quad (45)$$

## Identifying monetary policy shocks

where  $\varphi_{4,3}(1)$  is the impact multiplier that quantifies the consequence of a one unit increase in the error term  $\varepsilon_{3,t}$  in the equation of  $\Delta \log m_t$  for the value of  $\Delta r_t$ , all other innovations held constant. Figure 7 displays conditional forecasts.

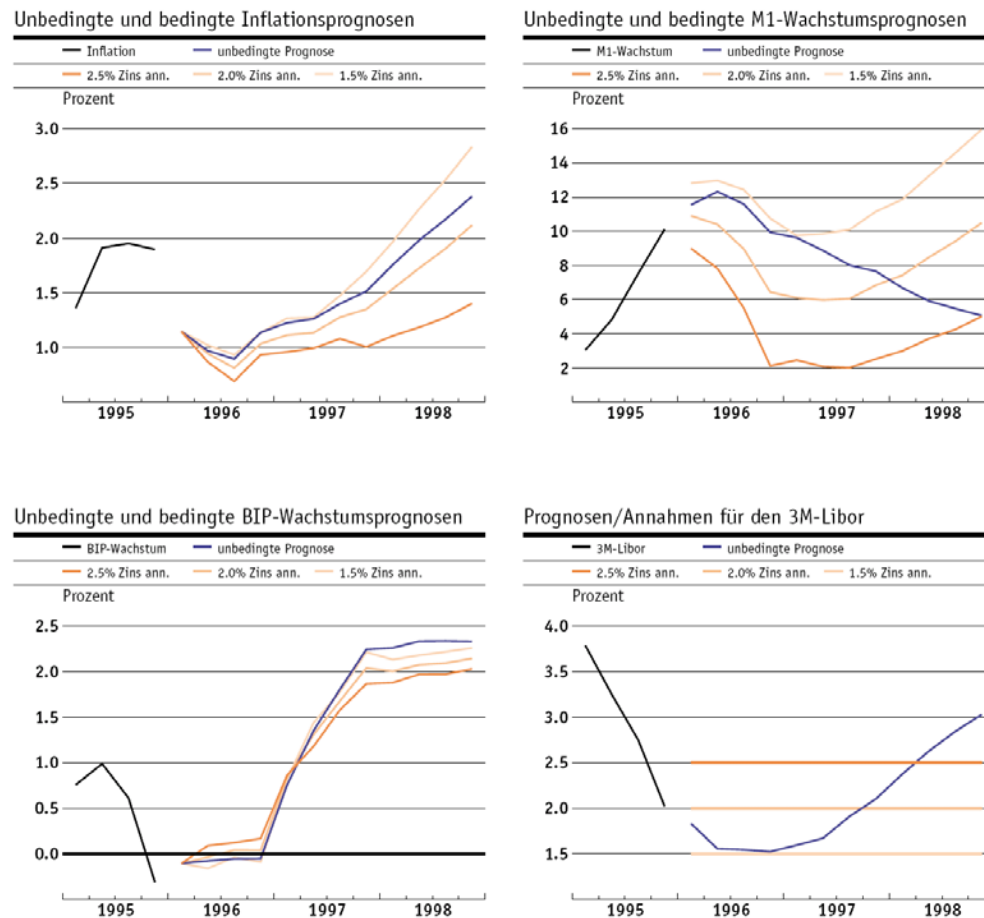


Figure 7: Conditional forecasts of the SNB SVAR.

## 5 Importance of the Ordering: Christiano et al. (1998)

Source: Christiano et al. (1998)

### 5.1 Methodical Results

Christiano et al. (1998) provide a far-reaching methodical result considering the effect of identifying restrictions (i.e. the ordering of the variables) on the shape of the impulse-response functions of a monetary policy shock.

Christiano et al. (1998) set up a SVAR following the notation in equation (29):

$$A_0 Z_t = C_0 + A_1 Z_{t-1} + A_2 Z_{t-2} + \dots + A_q Z_{t-q} + \varepsilon_t$$

The moving average coefficient matrices are defined as in equation (21):

## Identifying monetary policy shocks

$$\frac{\partial Z_{t+s}}{\partial u_t} = \Psi_s$$

The vector  $Z_t$  is partitioned into 3 distinct sets of variables:

$$Z_t = \begin{pmatrix} X_{1t} \\ (k_1 \times 1) \\ S_t \\ (1 \times 1) \\ X_{2t} \\ (k_2 \times 1) \end{pmatrix} \quad (46)$$

where  $X_{1t}$  is a partition of  $k_1$  variables that are not directly affected by monetary policy.  $S_t$  is the monetary policy instrument, and  $X_{2t}$  is a partition of  $k_2$  variables that are contemporaneously affected by monetary policy. We define  $k = k_1 + 1 + k_2$ . Note that the  $(k_1 + 1)$  th row of the SVAR specifies the monetary policy feedback rule as specified in section 1.2, where the information set  $I_t$  is captured by the variables in  $X_{1t}$ . I.e. monetary policy actions  $S_t$  are assumed to linearly depend on a set of variables  $X_{1t}$  that characterize the state of the economy:  $S_t = f(X_{1t}) + \sigma_s \varepsilon_t^s$ .

The central assumption made by Christiano et al. (1998) is that monetary policy shocks are orthogonal to the information set  $I_t$  (represented in  $X_{1t}$ ) which is considered by the monetary policy maker. This is the so called recursiveness assumption. By consequence matrix  $A_0$  which defines the contemporaneous relations between the endogenous variables has the following structure:

$$A_0 = \begin{pmatrix} a_{1,1} & 0 & 0 \\ (k_1 \times k_1) & (k_1 \times 1) & (k_1 \times k_2) \\ a_{2,1} & a_{2,2} & 0 \\ (1 \times k_1) & (1 \times 1) & (1 \times k_2) \\ a_{3,1} & a_{3,2} & a_{3,3} \\ (k_2 \times k_1) & (k_2 \times 1) & (k_2 \times k_2) \end{pmatrix} \quad (47)$$

Christiano et al. (1998) define 3 sets of solutions for  $A_0$ :

$$Q_V = \{A_0 : A_0(A_0^{-1})' = V\} \quad (48)$$

$$Q_\tau = \{A_0 : \tau \text{vec}(A_0) = 0\} \quad (49)$$

$$Q_S = \{A_0 : A_0 \text{ has strictly positive diagonal elements}\} \quad (50)$$

where

$\tau$  is a  $(l \times (k \times k))$  matrix that contains  $l$  restrictions of  $A_0$  as specified in (47), hence  $l = k_1 + k_1 * k_2 + k_2$

$\text{vec}(A_0)$  is a  $(k \times k)$  dimensional vector that is composed of the  $k$  columns of  $A_0$

$Q_V$  assumes restricts the variance-covariance matrix of structural shocks to be equal to the identity matrix  $I$ . Hence,  $V = A_0^{-1}D(A_0^{-1})' = A_0^{-1}(A_0^{-1})'$ . Of course,  $Q_V$  will contain many elements because  $A_0$  contains  $k \times k$  unknowns while  $V$  only contains  $(k + 1)k / 2$  distinct parameters.  $Q_\tau$  imposes the recursive structure on  $A_0$ , as

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specified in (47).  $Q_S$  further restricts the set of possible solutions by restricting the diagonal elements of  $A_0$  to strictly positive values.

Christiano et al. (1998) show that the recursiveness assumption is not sufficient to identify all elements of  $A_0$ . However, they demonstrate that each matrix  $A_0$  that is an element of  $Q_\tau$  and  $Q_V$  generates the same impulse response function for the elements of  $Z_t$  to a monetary policy shock  $\varepsilon_t^s$ , and that for  $A_0 \in Q_V \cap Q_\tau \cap Q_S$  the ordering of the variables in  $X_{1t}$  and  $X_{2t}$  does not affect the impulse response functions. In other words: even if the recursiveness assumption is not sufficient to identify the structural parameters of  $A_0$ , it is sufficient to identify the response of  $Z_t$  to a monetary policy shock, which again is independent of the ordering of the variables in  $X_{1t}$  and  $X_{2t}$ !

### 5.2 Proofs

More formally, Christiano et al. (1998, 16) proclaim three propositions:

- "(i) The set  $Q_\tau$  and  $Q_V$  is nonempty and contains more than one element
- (ii) The  $(k_1 + 1)$  th column of  $\Psi_s$ ,  $s = 0, 1, \dots$  is invariant to the choice of  $A_0 \in Q_V \cap Q_\tau$ .
- (iii) Restricting  $A_0 \in Q_V \cap Q_\tau$  to be lower triangular with positive diagonal terms, the  $(k_1 + 1)$  th column of  $\Psi_s$ ,  $s = 0, 1, \dots$  is invariant to the ordering of the elements in  $X_{1t}$  and  $X_{2t}$ ."

In this section we present the proofs for these conjunctures. The proofs closely follow Christiano et al. (1998) but are slightly more elaborate.

- Proof of proposition (i):

- Define

$$W = \begin{pmatrix} W_{1,1} & \mathbf{0} & \mathbf{0} \\ (k_1 x k_1) & (k_1 x 1) & (k_1 x k_2) \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ (1 x k_1) & (1 x 1) & (1 x k_2) \\ \mathbf{0} & \mathbf{0} & W_{3,3} \\ (k_2 x k_1) & (k_2 x 1) & (k_2 x k_2) \end{pmatrix} \quad (51)$$

where  $W_{1,1}$  and  $W_{3,3}$  are arbitrary orthonormal matrices.

$$Q_{\bar{A}_0} = \{A_0 : A_0 = W \bar{A}_0\}$$

where  $\bar{A}_0$  is an arbitrary matrix conformable to  $W$ .

- It can be shown that

$$Q_{\bar{A}_0} = Q_V \cap Q_\tau \quad (52)$$

where  $\bar{A}_0$  is an arbitrary element of  $Q_V \cap Q_\tau$ :

- It holds that:  $A_0 \in Q_{\bar{A}_0}$  implies  $A_0 \in Q_V \cap Q_\tau$ :

-  $A_0 \in Q_V$  because:  $\bar{A}_0 \in Q_V$ ,  $W W' = I$ .

-  $A_0 \in Q_\tau$  because:  $W$  is blockdiagonal as defined in (51).

- It holds that:  $A_0 \in Q_{\bar{A}_0}$ . Consider an arbitrary  $A_0 \in Q_V \cap Q_\tau$ :

- We define  $W = A_0 \bar{A}_0^{-1}$ .

-  $\bar{A}_0$  is invertible because:  $\bar{A}_0 \in Q_V$ .

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- $W$  is block-lower triangular because:  $A_0, \bar{A}_0 \in Q_\tau$ , i.e.  $\bar{A}_0$  and  $A_0$  are block-lower triangular
- $WW' = 1$  because:
 
$$A_0^{-1}A_0^{-1'} = V = \bar{A}_0^{-1}\bar{A}_0^{-1'}$$

$$WW' = A_0\bar{A}_0^{-1}\bar{A}_0^{-1'}A_0' = A_0VA_0' = A_0A_0^{-1}A_0^{-1'}A_0' = I$$
- Hence,  $W$  has the form as specified in (51). Thus,  $A_0 \in Q_{\bar{A}_0}$  and  $Q_{\bar{A}_0} = Q_V \cap Q_\tau$ .
- $Q_V \cap Q_\tau$  is nonempty, because  $V$  can always be factorized into two triangular matrices applying the Choleski decomposition  $V = A_0^{-1}A_0^{-1'}$ , see section 4.1.1.
- $Q_V \cap Q_\tau$  entails more than one element because:
  - Define  $\bar{A}_0^{-1}\bar{A}_0^{-1'} = V$ .
  - Define  $W \neq I$ , where  $W$  is constructed from an identity matrix  $I$  by interchanging two of either the first  $k_1$  rows or the last  $k_2$  rows of  $I$ . Note that by construction,  $WW' = I$ .
  - Above we have derived that  $W\bar{A}_0 \in Q_V \cap Q_\tau$ . Since  $W\bar{A}_0 \neq \bar{A}_0$ ,  $Q_V \cap Q_\tau$  entails more than one element.

### - Proof of proposition (ii):

- Consider two matrices  $A_0, \tilde{A}_0 \in Q_\tau \cap Q_V$
- Since (52) holds, there exists a matrix  $W$  as specified in (51) for which  $\tilde{A}_0 = WA_0$ .
- Inversion yields  $\tilde{A}_0^{-1} = A_0^{-1}W'$ , i.e.

$$\begin{pmatrix} \tilde{a}_{1,1} & 0 & 0 \\ (k_1xk_1) & (k_1x1) & (k_1xk_2) \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & 0 \\ (1xk_1) & (1x1) & (1xk_2) \\ \tilde{a}_{3,1} & \tilde{a}_{3,2} & \tilde{a}_{3,3} \\ (k_2xk_1) & (k_2x1) & (k_2xk_2) \end{pmatrix}^{-1} = \begin{pmatrix} a_{1,1} & 0 & 0 \\ (k_1xk_1) & (k_1x1) & (k_1xk_2) \\ a_{2,1} & a_{2,2} & 0 \\ (1xk_1) & (1x1) & (1xk_2) \\ a_{3,1} & a_{3,2} & a_{3,3} \\ (k_2xk_1) & (k_2x1) & (k_2xk_2) \end{pmatrix}^{-1} \begin{pmatrix} W'_{1,1} & 0 & 0 \\ (k_1xk_1) & (k_1x1) & (k_1xk_2) \\ 0 & 1 & 0 \\ (1xk_1) & (1x1) & (1xk_2) \\ 0 & 0 & W'_{3,3} \\ (k_2xk_1) & (k_2x1) & (k_2xk_2) \end{pmatrix}$$

Hence the  $(k_1 + 1)$  th column of  $\tilde{A}_0^{-1}$  and  $A_0^{-1}$  are identical. The impulse-response function bases on the relationship  $\frac{\partial Z_{t+s}}{\partial \varepsilon_t^s} = \Psi_s A_0^{-1}$  (33). Thus the response to the  $(k_1 + 1)$  th structural innovation, which is the monetary policy shock  $\varepsilon_t^s$ , is invariant to the choice of  $A_0 \in Q_V \cap Q_\tau$ .

### - Proof of proposition (iii):

- Define  $D \neq I$ , where  $D$  is constructed from an identity matrix  $I$  arbitrarily reordering the first  $k_1$  rows and the last  $k_2$  rows of  $I$ .
- Define  $\tilde{Z}_t = DZ_t$ . Hence,  $\tilde{Z}_t$  corresponds to  $Z_t$  with the variables in  $X_{1t}$  and  $X_{2t}$  reordered arbitrarily.
- Consider the reduced form VAR for  $Z_t$  as specified equation (9):

$$Z_t = B_1Z_{t-1} + B_2Z_{t-2} + \dots + B_qZ_{t-q} + u_t$$

where the relationships between the reduced form parameters and the structural form parameters are defined in (30) and (31). In particular it holds that  $B_i = A_0^{-1}A_i, i = 1 \dots q$ . The impulse-response function of  $Z_t$  to  $\varepsilon_t^s$  is characterized by  $A_0$ .

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- Define  $A_0$  as the lower-triangular matrix with positive diagonal elements obtained by the Choleski decomposition  $V = A_0^{-1}A_0^{-1}$ .

- Hence, the structural and reduced form VAR for  $\tilde{Z}_t$  are given by:

$$\begin{aligned} A_0 D' D Z_t &= A_1 D' D Z_{t-1} + \dots + D \varepsilon_t \\ D Z_t &= D A_1^{-1} A_1 D' D Z_{t-1} + \dots + D A_1^{-1} D \varepsilon_t \\ \tilde{Z}_t &= D B_1 D' \tilde{Z}_{t-1} + \dots + D A_1^{-1} D \varepsilon_t \end{aligned}$$

With the variance-covariance matrix of reduced form errors  $E(D A_0^{-1} D \varepsilon_t)(D A_0^{-1} D \varepsilon_t)' = D V D'$ . Note that  $D' D = I = D^{-1} D$ . The impulse-response function of  $\tilde{Z}_t$  to  $\varepsilon_t^s$  is characterized by  $A_0 D'$ . Note that in general,  $A_0 D'$  is not lower triangular.

- Define  $\tilde{A}_0 = A_0 D'$ , with

$$\tilde{A}_0 = \begin{pmatrix} \tilde{a}_{1,1} & 0 & 0 \\ (k_1 x k_1) & (k_1 x 1) & (k_1 x k_2) \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & 0 \\ (1 x k_1) & (1 x 1) & (1 x k_2) \\ \tilde{a}_{3,1} & \tilde{a}_{3,2} & \tilde{a}_{3,3} \\ (k_2 x k_1) & (k_2 x 1) & (k_2 x k_2) \end{pmatrix}$$

Where  $\tilde{a}_{1,1}$  and  $\tilde{a}_{3,3}$  have full rank but are not necessarily lower triangular.

- It exists a decomposition  $\tilde{a}_{1,1} = Q_1 R_1, \tilde{a}_{3,3} = Q_3 R_3$ , where  $Q_1$  and  $Q_2$  are square orthonormal matrices while  $R_1$  and  $R_2$  are lower triangular matrices with positive diagonal elements.<sup>4</sup>

- Define

$$W = \begin{pmatrix} Q_1' & 0 & 0 \\ (k_1 x k_1) & (k_1 x 1) & (k_1 x k_2) \\ 0 & 1 & 0 \\ (1 x k_1) & (1 x 1) & (1 x k_2) \\ 0 & 0 & Q_3' \\ (k_2 x k_1) & (k_2 x 1) & (k_2 x k_2) \end{pmatrix}$$

- It holds that

-  $W W' = I$  because:  $Q_1, Q_2$  are each orthonormal matrices

-  $(W \tilde{A}_0)^{-1} (W \tilde{A}_0)^{-1}' = D V D'$  because:

$$\begin{aligned} (W \tilde{A}_0)^{-1} (W \tilde{A}_0)^{-1}' &= \tilde{A}_0^{-1} W^{-1} W^{-1}' \tilde{A}_0^{-1}' \\ &= \tilde{A}_0^{-1} \tilde{A}_0^{-1}' = D A_0^{-1} A_0^{-1} D' = D V D' \end{aligned}$$

-  $W \tilde{A}_0$  is lower triangular with positive diagonal elements

- Since  $(W \tilde{A}_0)^{-1} = \tilde{A}_0^{-1} W'$ , the  $(k_1 + 1)$ th columns of  $(W \tilde{A}_0)^{-1}$  and  $\tilde{A}_0^{-1}$  are identical. The impulse-response function bases on the relationship  $\frac{\partial Z_{t+s}}{\partial \varepsilon_t^s} = \Psi_s A_0^{-1}$  (33). Thus the response to the  $(k_1 + 1)$ th structural innovation, which is the monetary policy shock  $\varepsilon_t^s$ , is invariant to ordering of the variables in  $X_{1t}$  and  $X_{2t}$  given that  $A_0$  is lower triangular with positive diagonal terms.

### 5.3 Selected Empirical Results

<sup>4</sup> See Christiano et al (1998, 18).

### Identifying monetary policy shocks

Christiano et al. (1998) consider three benchmark recursive identification schemes which correspond to different specifications of  $S_t = f(X_{1t}) + \sigma_s \varepsilon_t^s$ .

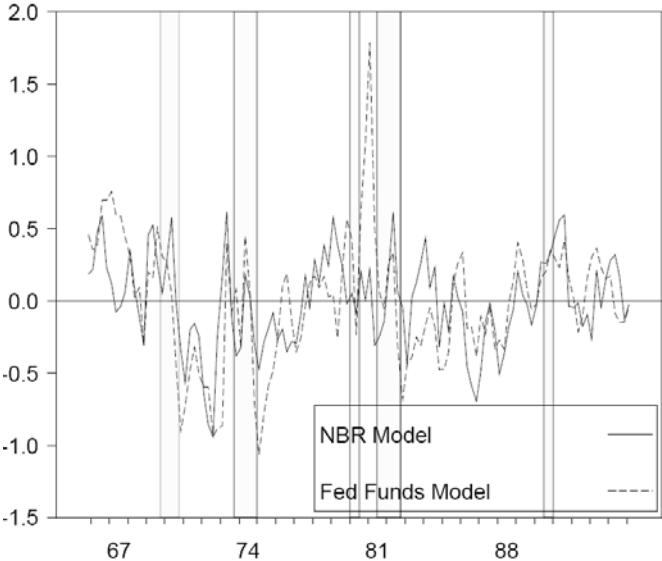
In the first identification scheme the Federal Funds rate (FF) is considered to be the policy instrument. It is assumed that the fed funds rate is contemporaneously affected by real GDP, the GDP deflator, and an index for commodity prices, while it has contemporaneous effects on the total reserves, the non-borrowed reserves and on a monetary aggregate (either M1 or M2; all variables in logs). Hence  $Z_t$  takes the form:

$$Z_t = (Y_t, P_t, PCOM_t, FF_t, TR_t, NBR_t, M_t)' \tag{53}$$

In a second identification scheme the non-borrowed reserves (NBR) are assumed to be the policy instrument. This is motivated by the empirical indication that innovations to non-borrowed reserves primarily reflect exogenous shocks to monetary policy, while innovations to monetary aggregates M1 and M2 primarily reflect shocks to money demand. Again it is assumed that the policy instrument is contemporaneously affected by real GDP, by the GDP deflator and by an index for commodity prices, while it has contemporaneous effects on the fed funds rate, on total reserves, and on a monetary aggregate (either M1 or M2; all variables in logs). Hence  $Z_t$  takes the form:

$$Z_t = (Y_t, P_t, PCOM_t, NBR_t, FF_t, TR_t, M_t)' \tag{54}$$

The estimated policy shocks in a VAR with 4 lags over the sample period Q3.1963 to Q2.1995 are displayed in the following graph:



**Figure 8: Monetary policy shocks, centered three quarter moving average, %. Source: Christiano et al. (1998).**

The figures display the centered three quarter moving average to reduce fluctuations in the series of serially uncorrelated monetary policy shocks. The shaded regions display business cycle peak and end at a trough. We see that the monetary policy shocks stemming from the model that takes the fed funds rate as policy instrument tend to be positive around the business cycle peak and tend to be negative around the time of the trough. I.e. policy is loose around the time of the trough and relatively tight around the peak.

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The impulse response functions of GDP and GDP deflator to a one standard deviation shock (annual standard deviation of the quarterly fed funds rate is 0.71%, of the non-borrowed reserves 1.53%) are displayed in Figure 9. See Christiano et al. (1998, 22) for a detailed description.

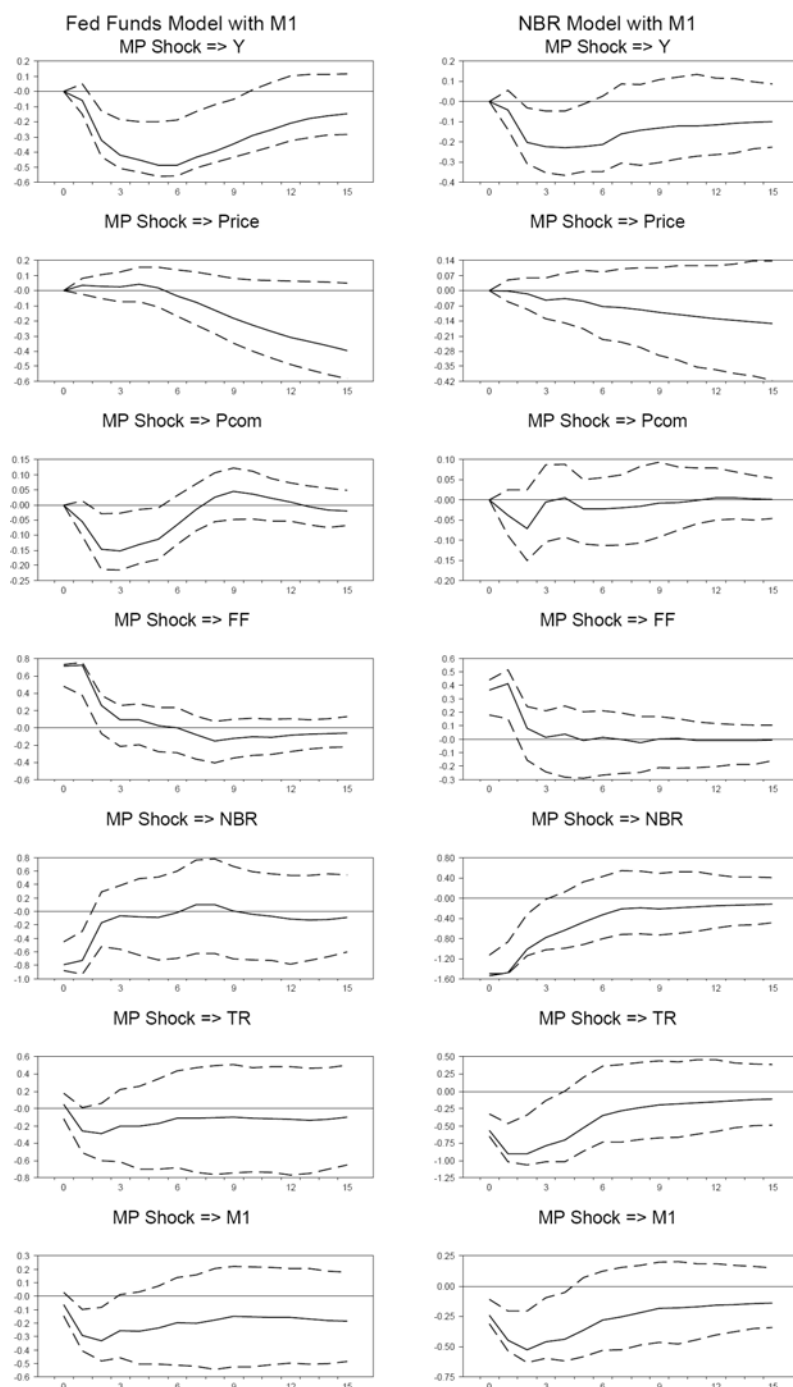


Figure 9: Impulse response functions. Source: Christiano et al. (1998).

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