I) Chapter: Complement to lecture 3.

II) Title: Ex Ante Real Rate and Natural Rate of Interest

III) Contents: Fisher equation, extracting the ex-ante real interest rate, estimating the natural rate of interest

IV) Documentation:

- Fama, Eugene F. (1975). Short-Term Interest Rates as Predictors of Inflation, *The American Economic Review*, Vol. 65, p. 269.
- Fama, Eugene F. and Michael R. Gibbons (1982). Inflation, Real Returns and Capital Investment, *Journal of Monetary Economics*, Vol. 9, p. 297.
- Garbade, Kenneth and Paul Wachtel (1978). Time Variation in the Relationship Between Inflation and Interest Rates, *Journal of Monetary Economics*, Vol. 4, p. 755
er Series, No. 82.
- Laubach, Thomas and John C. Williams (2003). Measuring the Natural Rate of Interest, *The Review of Economics and Statistics*, Vol. 85(4), p. 1063.

[- Stock, James H. and Mark W. Watson (1998). Median Unbiased Estimation of Coefficient Variance in a Time-Varying Parameter Model, *Journal of the American Statistical Association*, Vol. 93, p. 349.]

Table of contents:

1. Introduction

The state-space methodology has been applied to the real interest rate in several ways, of which we will cover two principal applications: The estimation of the ex ante real rate and the estimation of the natural rate of interest.

2. Estimating the ex ante real rate

2.1 The Fisher equation

Irving Fisher (1930) claimed that the one-period (nominal) interest rate R_t, observed at the end of period $t-1$ can be broken into an expected real rate of return r_t^e – or *ex-ante* real interest rate – for period t, and an expected rate of inflation π_t^e

$$R_t = r_t^e + \pi_t^e \tag{0.1}$$

or

$$\pi_t^e = -r_t^e + R_t \tag{0.2}$$

We cannot test if this relationship is true because we are unable to observe r_t^e and π_t^e directly. A solution would be to construct a proxy for the expected inflation and then to regress a nominal interest rate on the proxy and interpret the constant in this regression as the ex-ante real rate. If the coefficient on the expected inflation is found not to be significantly different from one, the Fisher relationship holds.

2.2 Fama

Such tests have been criticized because they assume a constant real rate over time and they jointly test the Fisher equation *and* the adequacy of the proxy for expected inflation. Fama (1975) proposed an alternative way of testing the Fisher hypothesis. By jointly testing the efficient market hypothesis and the Fisher equation, he finds a way around the unobservable expected inflation. Fama assumes that "if the market is efficient, then in setting the nominal price of a one-month [treasury] bill at $t-1$, it correctly uses all available information to asses the distribution of π_t^e". We can therefore decompose the actual inflation into expected inflation and an error term e_t with mean zero representing the errors of predicting inflation

$$\pi_t = E\left[\pi_t \middle| \Omega_{t-1}\right] + e_t = \pi_t^e + e_t \tag{0.3}$$

Using this result and substituting for π_t^e in (0.2), we get

$$\pi_t = -r_t^e + R_t + e_t \tag{0.4}$$

Fama further assumes that the ex-ante real interest rate is constant, which allows him to use OLS to run the regression

$$\pi_t = \alpha + \beta R_t + \varepsilon_t \tag{0.5}$$

where $\varepsilon_t \sim N\left(0, \sigma^2\right)$. Fama cannot reject the null hypothesis that $\beta = 1$ and he therefore concludes that the t-bill markets are efficient and that the Fisher hypothesis holds. He claims that his results support the assumption that r_t^e is constant over time.

2.3. A bi- and univariate TVP representation

Garbade and Wachtel (1978) have criticized Fama for assuming both α and β constant. They suggest a model where both coefficients are allowed to vary over time.

$$\pi_t = \alpha_t + \beta_t R_t + \varepsilon_t \tag{0.6}$$

where $\alpha_t = -r_t^e$ stands for the ex-ante real interest rate for period t, observed at the end of period $t-1$.
The model (0.6) can be cast into state-space form with observation equation

$$\pi_t = \mathbf{Z}_t \boldsymbol{\alpha}_t + \varepsilon_t \tag{0.7}$$

where $\mathbf{Z}_t = \begin{bmatrix} -1 & R_t \end{bmatrix}$, $\boldsymbol{\alpha}_t = \begin{bmatrix} r_t^e & \beta_t \end{bmatrix}'$ and $\varepsilon_t \sim N(0, \sigma^2)$ and the state equation

$$\boldsymbol{\alpha}_t = \mathbf{T}_t \boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t \tag{0.8}$$

where $\mathbf{T}_t = \mathbf{I}_2$, $\boldsymbol{\eta}_t = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}'$ and $\boldsymbol{\eta}_t \sim N(\mathbf{0}, \sigma^2 \mathbf{Q})$.
Garbade and Wachtel apply the Kalman fliter to this TVP model and compute smoothed estimates for r_t^e and β_t. Their results suggest that β_t is stationary over time, but r_t^e displays statistically significant intertemporal variation.
They proceed by imposing the restriction $\beta_t = 1$ and estimate the model with observation equation

$$r_t = r_t^e + \varepsilon_t \tag{0.9}$$

where $\varepsilon_t \sim N(0, \sigma^2)$. They used the definition of the *ex-post* real interest rate $r_t = R_t - \pi_t$ to get (0.9). The state equation is

$$r_t^e = r_{t-1}^e + \eta_t \tag{0.10}$$

with $\eta_t \sim N(0, \sigma^2 Q)$ where Q represents the signal-to-noise ratio.
Figure 1 shows Garbade and Wachtel's smoothed estimates of r_t^e.

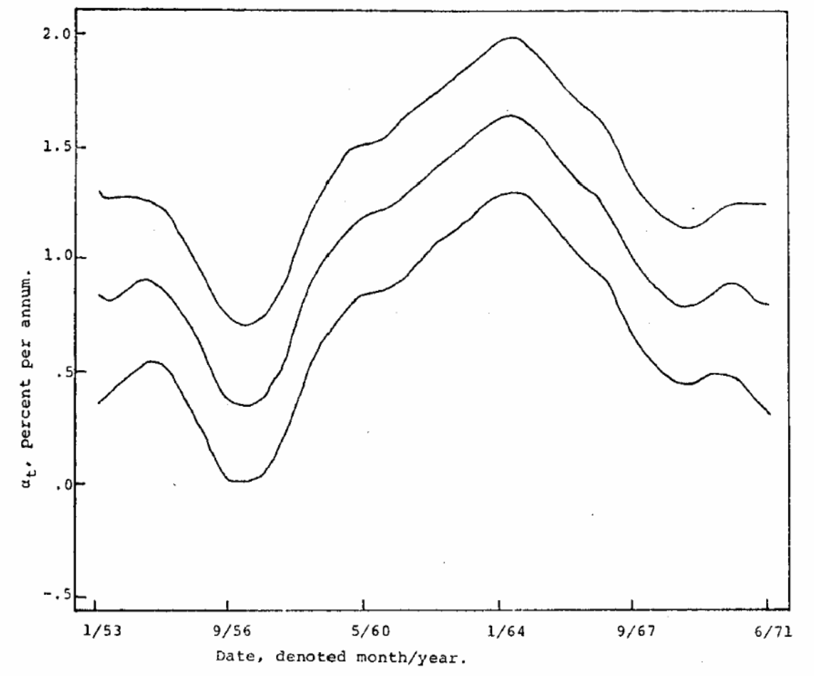


Figure 1: Smoothed estimated of ex-ante real rate; source Garbede Wachtel 1978

Since they cannot reject the hypothesis that $\beta_t = 1$, Grabade and Wachtel conclude "that there have not been even transient departures form an environment where an efficient inflationary premium is embedded in short term interest rates." Further, they conclude that the ex ante real rate of interest showed statistically significant variation.
Of course it is straight forward to use these results of the ex ante real interest rate to calculate the inflation expectations.

3. Estimating the natural rate of interest

3.1 Introduction

We will present the paper by Laubach and Williams 2003 (LW hereinafter) where they estimate the natural rate of interest for the USA. For a central bank it would be of great interest to exactly know where the natural rate of interest – the real rate of interest corresponding to a closed output gap and constant inflation – is and LW try to estimate it in a framework similar to the NAIRU estimation: They use some (ad hoc) reduced form of an IS-curve to relate the output gap to the real rate gap – the difference between ex ante real rate and the natural rate of interest r_t^*.

3.2 Empirical Framework

LW assume the natural rate of interest to follow the process

$$r_t^* = cg_t + z_t \tag{0.11}$$

where g_t denotes the trend growth rate and z_t stand for private households' preference shocks. LW use an IS and Phillips curve as observation equations

$$\tilde{y}_t = a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - r_{t-j}^*\right) + \varepsilon_{1,t} \tag{0.12}$$

$$\pi_t = B(L)\pi_{t-1} + b_{y,1}\tilde{y}_{t-1} + b_i\left(\pi_t^I - \pi_t\right) + b_o\left(\pi_t^O - \pi_t\right) + \varepsilon_{2,t} \tag{0.13}$$

where $\tilde{y}_t = 100*\left(y_t - y_t^*\right)$ denotes the output gap, r_{t-j} the ex ante real rate, $B(L)$ is a polynomial in the lag operator, π_t is the core price inflation, π_t^I the core import price inflation, π_t^O the crude imported oil price inflation and $\varepsilon_{k,t}$ are some white noise error terms. LW compute the ex ante real rate by proxy-ing expected inflation with inflation forecasts from an AR(3) model for inflation.
The state equations describe the processes of the taste shocks and the natural level of output y_t^*. For the taste shocks we have

$$z_t = D(L)z_{t-1} + \varepsilon_{3,t} \tag{0.14}$$

LW alternatively assume an AR(2) model or a random walk for the process of z_t. The natural output is modelled as a local linear trend with g_t being the trend growth rate.

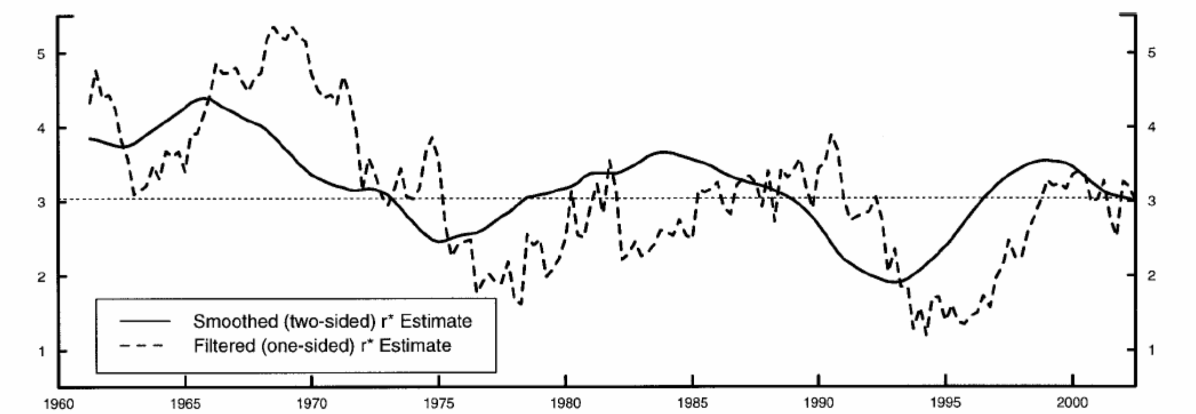
$$y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_{4,t}$$
$$g_t = g_{t-1} + \varepsilon_{5,t} \tag{0.15}$$

[Parenthesis: LW draw the reader's attention to the fact that σ_3 and σ_5 are likely to be biased towards zero due to the pile-up problem. This problem can arise when the state variable follows a random walk: If the signal-to-noise ratio of that state is small, then the MLE has its point mass at zero, i.e. it is estimated to be zero with probability 1.
Without going further into the details of how to solve the pile-up problem we just take note of its presence in this setup and trust LW that they dealt with it in the appropriate manner, namely by obtaining a priori estimates for the signal to noise ratios of the afflicted error terms. See Stock Watson 1998 for more details on the pile-up problem]

3.3 Estimation Results

Figure 1 shows the smoothed and filtered estimates of the natural interest rate.

FIGURE 1.—ONE- VS. TWO-SIDED ESTIMATES OF THE NATURAL RATE OF INTEREST (BASELINE MODEL)



The solid line shows the smoothed (two-sided) estimates of the natural rate of interest for the baseline specification in which z is assumed to follow a random walk. The dashed line shows the corresponding filter (one-sided) estimates.

We see considerable variation in the natural rate of interest over the past 40 years. The filtered estimates appear more volatile then the smoothed ones.
LW note that the estimated series have a large standard errors. They report sample average standard errors ranging from 2.78 to 0.9 percentage points in function of the different specifications.
"These results suggest that this source of uncertainty needs to be taken account of in analyzing monetary policies that feature responses to the natural rate of interest."