Exercises

Exercise 1:

Show the validity of the updating equation for $\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}$.

$$\left(\mathbf{X}_{t+1}^{'}\mathbf{X}_{t+1}\right)^{-1} = \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1} - \frac{\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}$$

Hint: Use thereby the Matrix Inversion Lemma (see exercice 3).

Exercise 2:

Show the validity of the updating equation for b.

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \mathbf{P}_t\mathbf{x}_{t+1}\frac{\left(y_{t+1} - \mathbf{x}_{t+1}^{'}\mathbf{b}_t\right)}{F_t}$$

Hint: Insert the updating equation for $\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}$ into the LS estimator for $\mathbf{b}_{t+1}$ and use the decomposition $\mathbf{X}_{t+1}^{'}\mathbf{y}_{t+1} = \mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}$.

Exercise 3:

The Matrix Inversion Lemma states that the following equality holds:

$$\left(\mathbf{A}+\mathbf{BCB}^{'}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1}+\mathbf{B}^{'}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{B}^{'}\mathbf{A}^{-1}$$

where A is $n \times n$, C is $m \times m$ and B is $n \times m$ is to be conformable. Convince yourself of the validity of the Lemma.

Solutions of the exercises:

Exercise 1:

Show the validity of the updating equation for $\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}$.

$$\left(\mathbf{X}_{t+1}^{'}\mathbf{X}_{t+1}\right)^{-1} = \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1} - \frac{\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}$$

Hint: Use thereby the Matrix Inversion Lemma (see exercice 3).

Solution:

Because of $\mathbf{X}_{t+1}^{'}\mathbf{X}_{t+1} = \mathbf{X}_t^{'}\mathbf{X}_t + \mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}$ we have:

$$\left(\mathbf{X}_t^{'}\mathbf{X}_t + \mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\right)^{-1} = \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1} - \frac{\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}$$

The equality follows immediately from the Matrix Inversion Lemma

$$\left(\mathbf{A}+\mathbf{BCB}^{'}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1}+\mathbf{B}^{'}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{B}^{'}\mathbf{A}^{-1}$$

with $\mathbf{A} = \mathbf{X}_t^{'}\mathbf{X}_t$, $\mathbf{B} = \mathbf{x}_{t+1}$, $\mathbf{C} = 1$.

Exercise 2:

Show the validity of the updating equation for b.

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \mathbf{P}_t\mathbf{x}_{t+1}\frac{\left(y_{t+1} - \mathbf{x}_{t+1}^{'}\mathbf{b}_t\right)}{F_t}$$

Hint: Insert the updating equation for $\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}$ into the LS estimator for $\mathbf{b}_{t+1}$ and use the decomposition $\mathbf{X}_{t+1}^{'}\mathbf{y}_{t+1} = \mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}$.

Solution

$$\mathbf{b}_{t+1} = \left(\mathbf{X}_{t+1}^{'}\mathbf{X}_{t+1}\right)^{-1}\mathbf{X}_{t+1}^{'}\mathbf{y}_{t+1}$$

$$\mathbf{b}_{t+1} = \left(\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1} - \frac{\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}\right)\left(\mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}\right)$$

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}y_{t+1} - \frac{\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}\left(\mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}\right)$$

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\left[y_{t+1} - \frac{\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}\left(\mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}\right)\right]$$

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\left[\frac{y_{t+1}\left(1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\right) - \mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\left(\mathbf{X}_t^{'}\mathbf{y}_t + \mathbf{x}_{t+1}y_{t+1}\right)}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}\right]$$

The terms $y_{t+1}\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}$ and $\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}y_{t+1}$ are equal because $\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}$ is a scalar. They cancel out. We therefore get:

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}\left[\frac{y_{t+1} - \mathbf{x}_{t+1}^{'}\mathbf{b}_t}{1+\mathbf{x}_{t+1}^{'}\left(\mathbf{X}_t^{'}\mathbf{X}_t\right)^{-1}\mathbf{x}_{t+1}}\right]$$

Exercise 3:
The Matrix Inversion Lemma states that the following equality holds:

$$\left(\mathbf{A}+\mathbf{BCB}^{'}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1}+\mathbf{B}^{'}\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{B}^{'}\mathbf{A}^{-1}$$

where **A** is $n\times n$, **C** is $m\times m$ and **B** is $n\times m$ is to be conformable. Convince yourself of the validity of the Lemma.

Solution
Let $\mathbf{D}$ be an $n\times n$ matrix defined by

$$\mathbf{D} = \left[\mathbf{A}+\mathbf{BCB}^{'}\right]^{-1}$$

where **A** and **C** are non-singular matrices of order *n* and *m* respectively, and **B** is $n\times m$.
Then

$$\mathbf{D} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left[\mathbf{C}^{-1}+\mathbf{B}^{'}\mathbf{A}^{-1}\mathbf{B}\right]^{-1}\mathbf{B}^{'}\mathbf{A}^{-1}$$

The result may be verified directly by showing that $\mathbf{DD}^{-1} = \mathbf{I}$. Let $\mathbf{E} = \mathbf{C}^{-1} + \mathbf{B'A}^{-1}\mathbf{B}$. Then

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} - \mathbf{BE}^{-1}\mathbf{B'A}^{-1} + \mathbf{BCB'A}^{-1} + \mathbf{BCB'A}^{-1}\mathbf{BE}^{-1}\mathbf{B'A}^{-1}$$
$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} + \left[-\mathbf{BE}^{-1} + \mathbf{BC} - \mathbf{BCB'A}^{-1}\mathbf{BE}^{-1}\right]\mathbf{B'A}^{-1}$$
$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} + \mathbf{BC}\left[-\mathbf{C}^{-1}\mathbf{E}^{-1} + \mathbf{I} - \mathbf{B'A}^{-1}\mathbf{BE}^{-1}\right]\mathbf{B'A}^{-1}$$
$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} + \mathbf{BC}\left[\mathbf{I} - \left[\mathbf{C}^{-1} + \mathbf{B'A}^{-1}\mathbf{B}\right]\mathbf{E}^{-1}\right]\mathbf{B'A}^{-1}$$

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} + \mathbf{BC}\left[\mathbf{I} - \mathbf{EE}^{-1}\right]\mathbf{B'A}^{-1} = \mathbf{I}$$

An important special case arises when **B** is an $n \times 1$ vector, **b**, and **C**=1. Then

$$\left(\mathbf{A} + \mathbf{bb'}\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{bb'A}^{-1}}{1 + \mathbf{b'A}^{-1}\mathbf{b}}.$$