Introduction to Bellman equations

Exercise 1: Growth

The representative consumer faces the following problem

$$\max_{\{c_t\}_0^\infty} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

subject to

$$y_t = f(k_t)$$
$$y_t = c_t + i_t$$
$$k_{t+1} = k_t(1-\delta) + i_t$$

The consumer has to distribute her income between consumption c_t and investment i_t in capital k_t which depreciates at rate $0 < \delta < 1$ and is the only input in the production function $f(\cdot)$.

1.) Which is/are the state variable(s)? Which is/are the control(s)?

Solution:
State: k_t
Controls: c_t and i_t, but note that the choice of one implicitly determines the other.

2.) Set up the Bellman equation!

Solution:
If we set it up to maximize with respect to tomorrow's state:

$$V(k) = \max_{k'} \{u(c)\} + \beta E_t [V(k')]$$
$$s.t. \quad c = f(k) - k' + k(1-\delta)$$

put together

$$V(k) = \max_{k'} \{u(f(k) - k' + k(1-\delta))\} + \beta E_t [V(k')]$$

3.) Derive the FOC(s), apply the envelope theorem and obtain the Euler equation(s). Interpret.

Solution:
The FOC is:

$$\frac{\partial V(k)}{\partial k'} = -u'(c) + \beta E_t [V(k')] = 0$$

By the envelope theorem we have:

$$\frac{\partial V(k)}{\partial k} = u(c)[f'(k) + (1-\delta)]$$

Shifting forward in time and substitute in the FOC

$$u'(c) = \beta u'(c')[f(k') + (1-\delta)]$$

We do have an Euler equation for consumption that looks similar to the usual one. The LHS shows the cost of reducing consumption today by 1 unit and the RHS shows the (discounted) extra utility gained through the investment of 1 more unit of capital.

Exercise 2: Durable goods consumption

The infinitely lived consumer faces the following problem

$$\max_{\{c_t\}_0^\infty} \left\{ \sum_{t=0}^{\infty} \beta^t E_t \left[u\left(c_t, d_t\right) \right] \right\} \tag{0.1}$$

subject to

$$\begin{aligned} a_{t+1} &= R_t \left(a_t + y_t - c_t - p_t e_t \right) \\ d_{t+1} &= d_t \left(1-\delta\right) + e_t \end{aligned} \tag{0.2}$$

The consumer's utility is in function of consumption of non-durable goods c_t and of the stock of durable goods d_t. As usual a_t denotes current wealth, y_t current income. The agent spends this on non-durable consumption c_t and on purchases e_t of the durable good at the price p_t. Durable goods depreciate with the rate $0 < \delta < 1$.

1.) Which is/are the state variable(s)? Which is/are the control(s)?

Solution:
The states are d_t and a_t.
The controls are c_t and e_t, but note that the choice of one implicitly determines the other.

2.) Set up the Bellman equation!

Solution:
If we set it up to maximize with respect to tomorrow's states:

$$V\left(a,d\right) = \max_{a',d'} \left\{ u\left(c,d\right) \right\} + \beta E\left[V\left(a',d'\right) \right]$$

$$s.t. \quad c = a + y - R^{-1} a' - p\left(d' - d\left(1-\delta\right)\right)$$

put together

$$V\left(a,d\right) = \max_{a',d'} \left\{ u\left(a + y - R^{-1} a' - p\left(d' - d\left(1-\delta\right)\right), d\right) \right\} + \beta E\left[V\left(a',d'\right) \right]$$

3.) Derive the FOC(s), apply the envelope theorem and obtain the Euler equation(s). Interpret.

Solution:
FOCs are:

$$\frac{\partial V\left(a,d\right)}{\partial a'} = -R^{-1} u_c\left(c,d\right) + \beta E\left[V_a\left(a',d'\right) \right] = 0$$

$$\frac{\partial V\left(a,d\right)}{\partial d'} = -p u_c\left(c,d\right) + \beta E\left[V_d\left(a',d'\right) \right] = 0$$

By the envelope theorem we have:

$$\frac{\partial V(a,d)}{\partial a} = u_c(c,d)$$
$$\frac{\partial V(a,d)}{\partial d} = p(1-\delta)u_c(c,d) + u_d(c,d)$$

Shifting forward in time and substitute in the first FOC

$$u_c(c,d) = \beta RE\left[V_a(a',d')\right] = \beta RE\left[u_c(c',d')\right]$$

gives the familiar Euler equation for (non-durable) consumption.

For the durable good we have

$$\begin{aligned} pu_c(c,d) &= \beta E\left[V_d(a',d')\right] \\ &= \beta E\left[p'(1-\delta)u_c(c',d') + u_d(c',d')\right] \end{aligned}$$

We see that marginal utility from non-durables consumption depends as well on the stock of durables.