I) Title: New Neoclassical Synthesis

II) Contents: New Keynesian Models

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See "REFERENCES" at the end of document.

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0. (Historical) Introduction

[Very closely following Goodfriend and King 1997, GK97 hereinafter]

0.1 The neoclassical synthesis

Around the 1950s and 1960s the IS-LM was the basic macroeconomic framework used by economists. This framework had Keynes' ideas of the determination of national income and neoclassical principles to guide microeconomic analysis.

In the beginning it was assumed that prices and wages were independent of real activity, but in the mid 1960s the economists had to change their mind and the Phillips curve was introduced, which then became the central part of policy analysis. Models were closed with wage and price sectors, indicating a trade-off between inflation and real activity. This trade-off became central to macroeconomic policy.

Monetary policy

Due to the short run price stickiness monetary policy was seen to have powerful effects on the real economy which suggested an activist aggregate demand management. "Monetary policy in the neoclassical synthesis was regarded as a powerful instrument, but one ill-suited to controlling inflation or to undertaking stabilization policy." (GK97) Credit channels and the long run interest rate were considered to be main transmission mechanisms.

0.2 Monetarism and Rational Expectations

Monetarism emerged in the 1960s as the intellectual descendant of pre-Keynesian quantity theory of money (Irving Fisher). Monetarists questioned the effectiveness of fiscal policy and the structural stability of the Phillips curve. According to monetarism, the source of business cycles lies in the short-run price stickiness. The basic framework was the quantity equation

$$\log Y_t = \log M_t + \log v_t$$

Money was considered to be the main transmission mechanism. Monetarists thought that most variation in interest rates comes from inflation premia.

Inflation and real activity

Monetarists had no short-run price equation but they attributed the short run non-neutrality of money to price level stickiness. Incomplete adjustment of expectations could lead wages and prices to respond sluggishly to changes in money. Sustained inflation should not affect the real activity in the long run (=a situation where expectations are correct).

Monetary policy

Monetarists suggested simple fixed rules for monetary policy. Since it should avoid monetary shocks to the economy, monetarists suggested rules for a constant growth of the quantity of money.

Rational expectatoins

The Lucas critique of macroeconomic policy introduced rational expectations in macroeconomics in the 1970s, questioning the neoclassical synthesis. Only misperceived monetary changes were supposed to have real effects.

Credibility
Lucas (1976): the effect of a shock cannot be calculated without understanding its persistence or the extent to which it was expected and prepared for in advance. One cannot predict the effect of a policy action at a point in time without taking account of the nature of the policy regime from which it comes. Sargent (1986): "inflation only seems to have a momentum of its own. It is actually the long-term government policy of persistently running large deficits and reacting money at high rates that imparts the momentum to the inflation rate."
The neoclassical synthesis had almost completely ignored the issue of credibility.

0.3 Real Business Cycles

With RBC one could (for the first time) compare alternative policies on the basis of utility or costs, rather than ad hoc objectives. One could finally analyze policies and shocks in a fully specified dynamic-stochastic system (as called for by rational expectations reasoning). RBC integrates the aspect of intertemporal substitution for consumption, investment and labour supply. In the real general equilibrium model intertemporal optimization of consumption and labour supply of households is combined with intertemporal analysis of investment and labour demand of profit maximizing firms. -> General equilibrium where prices and quantities are simultaneously determined.
Productivity shocks
The RBC model is capable of generating business cycles resembling those of an economy driven by Solow residuals. Productivity shocks raise or lower output as seen by Solow's decomposition

$$\frac{dy_t}{y_t} = \left(s_n \frac{dn_t}{n_t} + s_k \frac{dk_t}{k_t} \right) + \frac{da_t}{a_t}$$

and those shocks affect macroeconomic activity because they affect factor demand schedules. "These marginal (substitution) influences interact with the smoothing motivation built into households' preferences to govern the dynamic response of the economy." (GK97) -> Procyclicality of labour input and response of investment.
Money
Endogenous variations in money supply at least partly explain the correlation of money and output but not very well the cyclical variation in real and nominal interest rates. The predicted impact of changes in inflation expectations is quantitatively small in flexible prices models. Sustained inflation is found to be bad for welfare
Fiscal Policy
Taxes have important impact on real activity since they influence the marginal returns to capital and/or labour. Taxes work like productivity shocks

0.4 New Keynesian Economics

The New Keynesian models evolved in response to the monetarist controversy, Lucas critique and RBC flex-price framework.
First generation
Gordon (1982) estimated price equations of the form

$$\pi_t = \lambda(L)\pi_{t-1} + G(\log Y_t - \log Y_{t-1}) + ps_t + \eta_t$$

where π_t is inflation, $\lambda(L)$ a polynomial in the lag operator and ps_t are price shocks.

Gordon found a numerically small value of G. The lags in the price equation were estimated to be important. But he also found that the estimates changed from subperiod to subperiod of his sample (1892-1980).

Taylor (1980) suggested a model with staggered wage setting where the price level was determined by the average wage. Taylor made the monetarist assumption that nominal expenditure was determined by a quantity equation and he used an activist money stock rule. Rather than viewing the monetary policy to be the source of business-cycle impulses, Taylor viewed it as adjusting the money stock to the price level. Four implications: humped-shaped pattern of cyclical output, policy rule matters for the evolution of real activity, monetary policy trade-off between the variability of output and that of inflation, rational expectations mattered a great deal for the response of the economy to shocks and for the design of MP rules. The only major criticism as addressed at the unrealistic wage setting behaviour.

Second generation

In the second generation New Keynesian models the nominal stickiness comes from prices rather than wages. Imperfect competition and prices that are subject to costs of adjustment are used to model the impact of money on real output.

Monopolistic competition became an important concept to analyze how firms set prices. Firms are now modelled as setting the price using a markup over marginal cost. Sticky prices are motivated through the idea that there exists a cost of changing prices (menu costs). The combination of monopolistic competition and sticky prices is essential in order for money to have real effects on output: If demand goes up and prices remain unchanged, only in the monopolistic competition case the firm will raise output! Demand determines output. A firm will continue to satisfy demand as long as price>marginal costs. New Keynesian economics explains the procyclical movement in real wages and marginal cost: If firm wants to produce more, need more labour, higher wage, higher mc, but still ok as long as mc<p!

Monopolistic competition explains also natural level of unemployment due to market power of firms. A simple way to introduce monopolistic competition is to impose large fixed costs in the firms' production function. -> Constant marginal costs, diminishing average costs.

Dynamic price setting models

The first models for price dynamics based on costs of changing prices were introduced in the early 1970s. In those state-dependent models, firms decide on the timing and on the size of the price change in function of the current state of the economy. Since it is difficult to introduce state-dependent pricing in macroeconomic models, New Keynesian models have focused on time-dependent price adjustment rules. Firms are assumed to receive an exogenous signal which allows them to change their price. The prime example Calvo 1983 will be treated in detail later on.

0.5 The New (Neoclassical) Synthesis / The New Keynesian perspective

The New Neoclassical Synthesis (NNS) integrates the New Keynesian elements of imperfect competition and nominal rigidities into the dynamic general equilibrium framework coming from standard RBC literature.

The name New Neoclassical Synthesis comes from the fact that it inherits the spirit of the old "neoclassical" synthesis (paragraph 0.1). "The NNS models offer policy advice based on the idea that price stickiness implies that aggregate demand is a key determinant of real economic activity in the short run. [...] The central conclusion is that economic fluctuations cannot be interpreted or understood independently of monetary policy."
Compared to the neoclassical synthesis (paragraph 0.1) the equilibrium conditions in the NNS models are derived from intertemporal optimization at the household and firm level and the NNS is argued to have stronger theoretical foundations than traditional Keynesian models. The main idea is that temporal nominal price rigidities provide the key friction that gives rise to the non neutrality of monetary policy.

1. Households

1.1 Setup of the problem

The infinitely-lived representative consumer maximizes his utility

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right) \tag{0.1}$$

where E_0 is the expectation operator, β the utility discount factor, C_t consumption, $\frac{M_t}{P_t}$ real money balances and N_t hours worked. We assume C_t to be the CES aggregator of differentiated goods used à la Dixit and Stigliz (1977)

$$C_t = \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{0.2}$$

where $\varepsilon > 0$ is the intertemporal elasticity of substitution and $C_t(i)$ is the consumption of good i, which is one element of the continuum of goods $[0,1]$. The price of good i is $P_t(i)$ and the price of the composite good C_t is thus given by P_t.

The consumer faces the budget constraint

$$\int_0^1 P_t(i) C_t(i) di + M_t + (1+i_t)^{-1} B_t \leq M_{t-1} + B_{t-1} + W_t N_t \quad \text{for } t = 0, \ldots \infty \tag{0.3}$$

where $\int_0^1 P_t(i) C_t(i) di$ stands for expenditures on consumption of the differentiated goods, M_t for money holdings, B_t for investments in bonds and $W_t N_t$ for labour income.

Consumers are assumed to act like final goods producers who seek to minimize costs in order to "produce" one unit of the composite final good for consumption C_t. They face the cost minimization problem

$$\min_{C(i)} \int_0^1 P(i) C(i) di \quad s.t. \quad C = \left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{0.4}$$

We dropped the time index since the consumer faces the same problem at every moment in time. To solve this problem, form the Lagrangian

$$L = \int_0^1 P(i) C(i) di - P\left[\left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} - C\right] \tag{0.5}$$

where the Lagrange multiplier, which has the interpretation of being the marginal cost of consuming one more unit of C, has been replaced by P since this is by definition the price of one unit of C.

The FOC is

$$\frac{\partial L}{\partial C(i)} = P(i) - P\frac{\varepsilon}{\varepsilon-1}\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} C(i)^{\frac{\varepsilon-1}{\varepsilon}-1} = 0$$
$$= P(i) - P\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}} C(i)^{-\frac{1}{\varepsilon}} = 0 \qquad (0.6)$$

for all $i \in [0,1]$.

The demand for good $C(i)$ is thus

$$C(i) = \left(\frac{P(i)}{P}\right)^{-\varepsilon} \left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\frac{P(i)}{P}\right)^{-\varepsilon} C \qquad (0.7)$$

Rewriting the FOC and integrating over the continuum of goods, the general price index P can be expressed as

$$P(i)^{1-\varepsilon} = \left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{-1} P^{1-\varepsilon} C(i)^{\frac{\varepsilon-1}{\varepsilon}}$$
$$\int_0^1 P(i)^{1-\varepsilon} di = \left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{-1} P^{1-\varepsilon} \left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right) \qquad (0.8)$$
$$P = \left(\int_0^1 P(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$$

And an expression for $\int_0^1 P_t(i) C_t(i) di$ can be found by multiplying the FOC by $C(i)$ and integrating over the continuum of goods to get

$$P(i)C(i) = P\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}} C(i)^{1-\frac{1}{\varepsilon}}$$
$$\int_0^1 P(i)C(i) di = P\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}} \left(\int_0^1 C(i)^{1-\frac{1}{\varepsilon}} di\right) \qquad (0.9)$$
$$\int_0^1 P(i)C(i) di = P\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}+1} = P\left(\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = PC$$

Substituting this result in the budget constraint, we get

$$P_t C_t + M_t + (1+i_t)^{-1} B_t \leq M_{t-1} + B_{t-1} + W_t N_t \qquad (0.10)$$

1.2 Optimization

The consumer's problem is therefore characterized by

$$\max_{\{C_t\}_0^\infty,\left\{\frac{M_t}{P_t}\right\}_0^\infty,\{N_t\}_0^\infty}\left\{E_0\sum_{t=0}^{\infty}\beta^t U\left(C_t,\frac{M_t}{P_t},N_t\right)\right\} \tag{0.11}$$

subject to

$$C_t=\frac{M_{t-1}}{P_t}+\frac{B_{t-1}}{P_t}+\frac{W_t}{P_t}N_t-\frac{M_t}{P_t}-\left(1+i_t\right)^{-1}\frac{B_t}{P_t} \tag{0.12}$$

for $t=1,\ldots,\infty$. Note that we replaced the $\leq$ by the equality sign.
We want to solve this problem by using Bellman's equation

$$V_t\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)=\max_{C_t,\frac{M_t}{P_t},N_t}U\left(C_t,\frac{M_t}{P_t},N_t\right)+\beta E_t V_{t+1}\left(\frac{M_t}{P_{t+1}},B_t\right) \tag{0.13}$$

subject to (0.12). The state variables are $\frac{M_{t-1}}{P_t}$ and B_{t-1} and the choice variables are $C_t,\frac{M_t}{P_t}$ and N_t. The first order conditions for (0.13) are

$$\frac{\partial V\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)}{\partial B_t}=-\left(1+i_t\right)^{-1}P_t^{-1}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)+\beta E_t V_{B_t}\left(\frac{M_t}{P_{t+1}},B_t\right)=0 \tag{0.14}$$

$$\frac{\partial V\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)}{\partial N_t}=\frac{W_t}{P_t}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)+U_{N_t}\left(C_t,\frac{M_t}{P_t},N_t\right)=0 \tag{0.15}$$

$$\frac{\partial V\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)}{\partial\frac{M_t}{P_t}}=-U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)+U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)+\beta E_t\frac{P_t}{P_{t+1}}V_{\frac{M_t}{P_{t+1}}}\left(\frac{M_t}{P_{t+1}},B_t\right)=0 \tag{0.16}$$

Note that we have used the trick

$$V_{t+1}\left(\frac{M_t}{P_{t+1}},B_t\right)=V_{t+1}\left(\frac{P_t}{P_{t+1}}\frac{M_t}{P_t},B_t\right)$$

in order to get the expression (0.16). From the envelope theorem we have

$$\frac{\partial V\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)}{\partial B_{t-1}}=P_t^{-1}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right) \tag{0.17}$$

$$\frac{\partial V\left(\frac{M_{t-1}}{P_t},B_{t-1}\right)}{\partial\frac{M_{t-1}}{P_t}}=U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right) \tag{0.18}$$

Solving for the *labour supply* is straightforward, from (0.15) we get

$$
-\frac{U_{N_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}{U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}=\frac{W_t}{P_t} \tag{0.19}
$$

To get the *Euler equation for consumption* we shift (0.17) forward in time by 1 unit and substitute in (0.14)

$$
\begin{aligned}
&(1+i_t)^{-1}P_t^{-1}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)=\beta E_t V_{B_t}\left(\frac{M_t}{P_{t+1}},B_t\right)\\
&(1+i_t)^{-1}P_t^{-1}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)=\beta E_t P_{t+1}^{-1}U_{C_{t+1}}\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}},N_{t+1}\right)\\
&U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)=\beta(1+i_t)E_t\frac{P_t}{P_{t+1}}U_{C_{t+1}}\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}},N_{t+1}\right)
\end{aligned} \tag{0.20}
$$

To get the *money demand* we shift (0.18) forward in time by 1 unit, substitute it in (0.16) and use the Euler equation to get

$$
\begin{aligned}
&U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)=U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)-\beta E_t\frac{P_t}{P_{t+1}}V_{\frac{M_t}{P_{t+1}}}\left(\frac{M_t}{P_{t+1}},B_t\right)\\
&U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)=U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)-\beta E_t\frac{P_t}{P_{t+1}}U_{C_{t+1}}\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}},N_{t+1}\right)\\
&U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)=U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)-(1+i_t)^{-1}U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)\\
&\frac{U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)}{U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}=\left(\frac{i_t}{1+i_t}\right)
\end{aligned} \tag{0.21}
$$

At this stage we make a specific assumption about the functional form of the utility function. Let

$$
U\left(C_t,\frac{M_t}{P_t},N_t\right)=\frac{C_t^{1-\sigma}}{1-\sigma}+\frac{(M_t/P_t)^{1-\nu}}{1-\nu}-\frac{N_t^{1+\varphi}}{1+\varphi} \tag{0.22}
$$

The *labour supply* is

$$
\begin{aligned}
&\frac{W_t}{P_t}=-\frac{U_{N_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}{U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}\\
&\frac{W_t}{P_t}=\frac{N_t^{\varphi}}{C_t^{-\sigma}}
\end{aligned} \tag{0.23}
$$

The *Euler equation* is

$$\begin{aligned} U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)&=\beta\left(1+i_t\right)E_tU_{C_{t+1}}\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}},N_{t+1}\right)\frac{P_t}{P_{t+1}} \\ C_t^{-\sigma}&=\beta\left(1+i_t\right)E_tC_{t+1}^{-\sigma}\frac{P_t}{P_{t+1}} \end{aligned} \tag{0.24}$$

The *money demand* is

$$\begin{aligned} \frac{U_{\frac{M_t}{P_t}}\left(C_t,\frac{M_t}{P_t},N_t\right)}{U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}&=\frac{i_t}{1+i_t} \\ \frac{\left(M_t/P_t\right)^{-\nu}}{C_t^{-\sigma}}&=\frac{i_t}{1+i_t} \\ \frac{M_t}{P_t}&=C_t^{\frac{\sigma}{\nu}}\left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{\nu}} \end{aligned} \tag{0.25}$$

2. Firms

2.1 Firms with flexible prices

We assume monopolistic competition among a continuum of firms, where firm $i \in [0,1]$ produces good i according to the CRTS production function

$$Y_t(i) = A_t N_t(i) \tag{0.26}$$

The log of technology A_t is assumed to follow an exogenous difference-stationary stochastic process

$$\Delta a_t = \rho_a \Delta a_{t-1} + \varepsilon_t^a$$

where ε_t^a is white noise and $\rho_a \in [0,1)$.

From the consumers cost minimization exercise, we know that firm i faces demand

$$Y_t(i) = C_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \tag{0.27}$$

for its output and is thus confronted with the problem of maximizing

$$\max_{P_t(i)} P_t(i) Y_t(i) - TC\left(W_t, Y_t(i)\right) \tag{0.28}$$

subject to (0.27), where $TC\left(W_t, Y_t(i)\right)$ denotes the total cost of producing good i in function of wages and output

$$TC\left(W_t, Y_t(i)\right) = W_t N_t = W_t \frac{Y_t(i)}{A_t}$$

The FOC yields

$$Y_t(i) + P_t(i) \frac{\partial Y_t(i)}{\partial P_t(i)} - MC_t^n(i) \frac{\partial Y_t(i)}{\partial P_t(i)} = 0 \tag{0.29}$$

where the nominal marginal cost for good i is given by

$$MC_t^n(i) = \frac{\partial TC\left(W_t, Y_t(i)\right)}{\partial Y_t(i)} = \frac{W_t}{A_t} \tag{0.30}$$

Note that $MC_t^n(i) = MC_t^n \ \forall \mathrm{i}$ since all firms are using the same technology A_t and pay the same wages W_t.

We can rewrite the FOC in function of the price elasticity of demand

$$\frac{\partial Y_t(i)}{\partial P_t(i)} \frac{P_t(i)}{Y_t(i)} = -\varepsilon$$

by premultiplying (0.29) by $\dfrac{\partial P_t(i)}{\partial Y_t(i)} \dfrac{1}{P_t(i)}$ to get

$$\frac{Y_t(i)}{P_t(i)} \frac{\partial P_t(i)}{\partial Y_t(i)} + 1 = \frac{MC_t^n}{P_t(i)}$$

$$\left(-\frac{1}{\varepsilon} + 1 \right) = \frac{MC_t^n}{P_t(i)} \tag{0.31}$$

$$P_t(i) = MC_t^n \left(\frac{\varepsilon}{\varepsilon - 1} \right)$$

This expression reproduces the standard result for monopolistic competition, namely that producers set a price above marginal cost, i.e. they use a markup $\frac{\varepsilon}{\varepsilon-1}$. Expression (0.31) also states that all producers set the same price $P_t = MC_t^n\left(\frac{\varepsilon}{\varepsilon-1}\right)$. The real marginal cost is therefore

$$MC_t = \frac{\varepsilon-1}{\varepsilon} \tag{0.32}$$

We can obtain an implicit formulation of labour demand by combining (0.30)-(0.32)

$$\begin{aligned} P_t &= \frac{W_t}{A_t}\left(\frac{\varepsilon}{\varepsilon-1}\right) \\ \frac{W_t}{P_t} &= A_t MC_t \end{aligned} \tag{0.33}$$

2.2 Price rigidity through staggered pricing

2.2.1 Setup

We introduce price stickiness in order to model an economy with rigidities. The approach chosen is based on the idea of Guillermo Calvo (1983).
The basic idea is that firms can change – i.e. reset – the price they charge for their products only at different moments in time. Calvo assumes that a firm can reset its price only when it receives a signal to do so (this is sometimes called "the Calvo-fairy coming around"). The probability of receiving such a signal in period h from now is modelled by the Poisson distribution. Recall the density function [for something to happen x-times in a time interval which on average happens λ times] is given by

$$f(x)=\frac{e^{-\lambda}\lambda^{x}}{x!}$$

Setting $\lambda=(1-\theta)$ (for $0\le\theta\le1$) as the probability that the Calvo-fairy visits firm i, the probability of the fairy visiting the firm once, namely in period h from now, is given by

$$e^{-(1-\theta)}\cdots e^{-(1-\theta)}(1-\theta)e^{-(1-\theta)}=(1-\theta)e^{-(1-\theta)h}$$

This probability is assumed to be independent of other firms resetting their prices and of the time elapsed since the last price reset. In discrete time the equivalent expression is

$$(1-\theta)\theta^{h-1}$$

Note that the average duration of a price is given by

$$\sum_{t=1}^{\infty}t(1-\theta)\theta^{t-1}=(1-\theta)+2(1-\theta)\theta+3(1-\theta)\theta^{2}+4(1-\theta)\theta^{3}+\ldots=\frac{1}{(1-\theta)}$$

This approach is very convenient due to the fact that "since there is a continuum of firms, we can appeal to the 'law of large numbers' to deduce that a number of $(1-\theta)$ firms (also a continuum) will receive the price-change signal per unit of time" (Calvo 1983). Instead of having to aggregate the behaviour of many different firms we can just assume on an economy wide level that a fraction $(1-\theta)$ of firms are allowed to change their prices and a fraction θ cannot change the price.
The general price index P_t (0.8) today is therefore a sum (going back in time infinitely) of prices previously set by firms which could reset their price in the past.

$$P_t=\left(\int_0^1 P^*(i)^{1-\varepsilon}\,di\right)^{\frac{1}{1-\varepsilon}}$$

$$P_t=\left[(1-\theta)P_t^{*1-\varepsilon}+(1-\theta)\theta P_{t-1}^{*\;1-\varepsilon}+(1-\theta)\theta^{2}P_{t-2}^{*\;1-\varepsilon}+\ldots\right]^{\frac{1}{1-\varepsilon}}\qquad(0.34)$$

$$P_t=\left[(1-\theta)\sum_{j=0}^{\infty}\theta^{j}P_{t-j}^{*\;1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

where P_{t-j}^{*} denotes the optimal reset price j periods back in the past. Note that when going from the first to the second line in (0.34), we implicitly assumed that all firms, which are allowed to change prices at a given moment in time, choose the same reset price, hence we dropped the i-index. This symmetry of firms' decisions will be proven to hold later. We can rewrite (0.34) as a non-linear difference equation

$$
\begin{aligned}
P_t &= \left[(1-\theta) \sum_{j=0}^{\infty} \theta^j {P_{t-j}^*}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \\
&= \left[(1-\theta) P_t^{*1-\varepsilon} + (1-\theta)\theta {P_{t-1}^*}^{1-\varepsilon} + (1-\theta)\theta^2 {P_{t-2}^*}^{1-\varepsilon} + \ldots \right]^{\frac{1}{1-\varepsilon}} \\
&= \left[(1-\theta) P_t^{*1-\varepsilon} + (1-\theta) \sum_{j=1}^{\infty} \theta^j {P_{t-j}^*}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \\
&= \left[(1-\theta) P_t^{*1-\varepsilon} + \theta \left\{ \left[(1-\theta) \sum_{j=0}^{\infty} \theta^j {P_{t-1-j}^*}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \right\}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \\
&= \left[(1-\theta) P_t^{*1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
\tag{0.35}
$$

We see that the current price level is a weighted average of the newly reset prices and of the price level of the previous period.
Let us now study how the individual firm determines the optimal reset price. When a firm can reset the price for its good, it has to choose the optimal reset price $P_t^*(i)$ that maximizes its market value

$$
\Omega_t = \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} \left[P_t^*(i) Y_{t+k}(i) - TC\left(W_{t+k}, Y_{t+k}(i)\right) \right] \right\} \tag{0.36}
$$

subject to the demand for good i

$$
Y_{t+k}(i) = C_{t+k} \left(\frac{P_t^*(i)}{P_{t+k}} \right)^{-\varepsilon} \tag{0.37}
$$

where $Q_{t,t+k}$ denotes a stochastic discount factor.

2.2.2 Optimization

Maximization yields the FOC

$$
\frac{\partial \Omega_t}{\partial P_t^*(i)} = \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} \left[Y_{t+k}(i) + P_t^*(i) \frac{\partial Y_{t+k}(i)}{\partial P_t^*(i)} - MC_{t+k}^n \frac{\partial Y_{t+k}(i)}{\partial P_t^*(i)} \right] \right\} = 0 \tag{0.38}
$$

Using the price elasticity of demand for the case when $P_t^*(i)$ remains fixed for k periods

$$
\frac{\partial Y_{t+k}(i)}{\partial P_t^*(i)} \frac{P_t^*(i)}{Y_{t+k}(i)} = -\varepsilon
$$

we can rewrite the FOC

$$
\begin{aligned}
0 &= \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} \left[Y_{t+k}(i) - \varepsilon Y_{t+k}(i) + MC_{t+k} P_{t+k} \frac{Y_{t+k}(i)}{P_t^*(i)} \varepsilon \right] \right\} \\
0 &= \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} Y_{t+k}(i) \left[1 - \varepsilon + MC_{t+k} \frac{P_{t+k}}{P_t^*(i)} \varepsilon \right] \right\}
\end{aligned}
\tag{0.39}
$$

or equivalently

$$(\varepsilon-1)\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}(i)\right\}=\frac{\varepsilon}{P_t^*(i)}\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}(i)MC_{t+k}^n\right\}$$

$$P_t^*(i)=\frac{\varepsilon}{\varepsilon-1}\frac{\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}(i)MC_{t+k}^n\right\}}{\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}Y_{t+k}(i)\right\}} \tag{0.40}$$

There are two things to note from the *optimal price setting rule* (0.40).
First, note that if $\theta=0$, we are back in the flexible price case with

$$P_t^*(i)=\frac{\varepsilon}{\varepsilon-1}MC_t^n$$

[Substituting $\theta=0$ in (0.40) implies that all terms of the sums are multiplied by zero, except the first since $0^0\equiv1$. The expectations operator drops since we are in time t.]
Second, the optimal price setting rule implies that due to symmetry, all firms that can reset their price will choose the same price. To see this, substitute the expression (0.37) for $Y_{t+k}(i)$ in (0.39) and multiply by $\frac{P_t^*(i)}{P_{t-1}}\frac{1}{1-\varepsilon}$

$$0=\sum_{k=0}^{\infty}\theta^k E_t\left\{Q_{t,t+k}C_{t+k}\left(\frac{P_t^*(i)}{P_{t+k}}\right)^{-\varepsilon}\left[\frac{P_t^*(i)}{P_{t-1}}+\frac{\varepsilon}{1-\varepsilon}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right]\right\} \tag{0.41}$$

From now on we can drop the (i) in the developments. The consumption Euler equation determines the stochastic discount factor

$$Q_{t,t+k}=(1+i_t)^{-k}=\beta^k\frac{U_{C_{t+k}}\left(C_{t+k},\frac{M_{t+k}}{P_{t+k}},N_{t+k}\right)}{U_{C_t}\left(C_t,\frac{M_t}{P_t},N_t\right)}\frac{P_t}{P_{t+k}} \tag{0.42}$$

$$Q_{t,t+k}=\beta^k\frac{C_{t+k}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+k}}$$

Substitute for the discount factor in (0.41)

$$0=\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{\frac{C_{t+k}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+k}}C_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}\left[\frac{P_t^*}{P_{t-1}}+\frac{\varepsilon}{1-\varepsilon}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right]\right\}$$

$$0=C_t^{\sigma}P_t^{*-\varepsilon}\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{P_tC_{t+k}^{1-\sigma}P_{t+k}{}^{\varepsilon-1}\left[\frac{P_t^*}{P_{t-1}}-\frac{\varepsilon}{\varepsilon-1}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right]\right\} \tag{0.43}$$

Get rid of $C_t^{\sigma}P_t^{*-\varepsilon}$, multiply by $P_t^{-\varepsilon}$ and substitute for $C_{t+k}=Y_{t+k}$ (implied by the market clearing condition)

$$0=\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{Y_{t+k}^{1-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}\left[\frac{P_t^*}{P_{t-1}}-\frac{\varepsilon}{\varepsilon-1}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right]\right\} \tag{0.44}$$

<u>2.2.3 Linearization around the steady state</u>

We will continue the development by linearizing the optimal pricing rule around a perfect foresight steady-state where we have

$$\frac{P_t^*}{P_{t-1}} = \Pi_{t-1,t}^*,\ \frac{P_{t+k}}{P_{t-1}} = \Pi_{t-1,t+k},\ Y_{t+k} = Y \text{ and } \frac{\varepsilon-1}{\varepsilon} = MC_{t+k}$$

Rewrite (0.44)

$$\sum_{k=0}^{\infty}(\theta\beta)^k \Pi_{t-1,t}^* E_t\left\{Y_{t+k}^{1-\sigma}\Pi_{t,t+k}^{\varepsilon-1}\right\} = \sum_{k=0}^{\infty}(\theta\beta)^k \frac{\varepsilon}{\varepsilon-1} E_t\left\{Y_{t+k}^{1-\sigma}\Pi_{t,t+k}^{\varepsilon-1}MC_{t+k}\Pi_{t-1,t+k}\right\} \tag{0.45}$$

and consider taking the first order Taylor approximation around the steady state. For notational ease, we will just show the developments of the $k-th$ term in the sums in equation (0.45)

$$\begin{aligned}
&Y^{1-\sigma}\Pi^{\varepsilon-1}\left(\Pi_{t-1,t}^* - \Pi^*\right) + (\varepsilon-1)\Pi^* Y^{1-\sigma}\Pi^{\varepsilon-2}\left(\Pi_{t,t+k} - \Pi\right) \\
&+(1-\sigma)\Pi^* Y^{-\sigma}\Pi^{\varepsilon-1}\left(Y_{t+k} - Y\right) \\
&= \frac{\varepsilon}{\varepsilon-1}(\varepsilon-1)Y^{1-\sigma}\Pi^{\varepsilon-2}MC\Pi\left(\Pi_{t,t+k}-\Pi\right) + \frac{\varepsilon}{\varepsilon-1}(1-\sigma)Y^{-\sigma}\Pi^{\varepsilon-1}MC\Pi\left(Y_{t+k}-Y\right) \\
&+\frac{\varepsilon}{\varepsilon-1}Y^{1-\sigma}\Pi^{\varepsilon-1}\Pi\left(MC_{t+k}-MC\right) + \frac{\varepsilon}{\varepsilon-1}Y^{1-\sigma}\Pi^{\varepsilon-1}MC\left(\Pi_{t-1,t+k}-\Pi\right)
\end{aligned}$$

which we can rewrite

$$\begin{aligned}
&\Pi^* Y^{1-\sigma}\Pi^{\varepsilon-1}\left[\frac{\Pi_{t-1,t}^* - \Pi^*}{\Pi^*} + (\varepsilon-1)\frac{\Pi_{t,t+k}-\Pi}{\Pi} + (1-\sigma)\frac{Y_{t+k}-Y}{Y}\right] \\
&= \frac{\varepsilon}{\varepsilon-1}Y^{1-\sigma}\Pi^{\varepsilon-1}MC\Pi\left[(\varepsilon-1)\frac{\Pi_{t,t+k}-\Pi}{\Pi} + (1-\sigma)\frac{Y_{t+k}-Y}{Y} + \frac{MC_{t+k}-MC}{MC} + \frac{\Pi_{t-1,t+k}-\Pi}{\Pi}\right]
\end{aligned}$$

Note that at the steady state $\Pi^* Y^{1-\sigma}\Pi^{\varepsilon-1} = \frac{\varepsilon}{\varepsilon-1}Y^{1-\sigma}\Pi^{\varepsilon-1}MC\Pi$, so we get after cancellations

$$\hat{\Pi}_{t-1,t}^* = M\hat{C}_{t+k} + \hat{\Pi}_{t-1,t+k}$$

where $\frac{\Pi_{t-1,t}^* - \Pi^*}{\Pi^*} = \hat{\Pi}_{t-1,t}^*$, $M\hat{C}_{t+k} = \frac{MC_{t+k}-MC}{MC}$ and $\hat{\Pi}_{t-1,t+k} = \frac{\Pi_{t-1,t+k}-\Pi}{\Pi}$ denote percentage deviations from steady state. Writing this solution for the sums, we get

$$\begin{aligned}
\sum_{k=0}^{\infty}(\theta\beta)^k \hat{\Pi}_{t-1,t}^* &= \sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{M\hat{C}_{t+k} + \hat{\Pi}_{t-1,t+k}\right\} \\
\hat{\Pi}_{t-1,t}^* &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{M\hat{C}_{t+k} + \hat{\Pi}_{t-1,t+k}\right\} \\
\hat{P}_t^* - \hat{P}_{t-1} &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{M\hat{C}_{t+k} + \hat{P}_{t+k} - \hat{P}_{t-1}\right\} \\
\hat{P}_t^* &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{M\hat{C}_{t+k} + \hat{P}_{t+k}\right\}
\end{aligned} \tag{0.46}$$

Now we are already very close to the New Phillips curve. Continue from the second last line of (0.46)

$$
\begin{aligned}
\hat{P}_t^* - \hat{P}_{t-1} &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left\{M\hat{C}_{t+k} + \hat{P}_{t+k} - \hat{P}_{t-1}\right\} \\
&= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(M\hat{C}_{t+k} + \sum_{h=0}^{k}\hat{\Pi}_{t+h}\right) \\
&= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(M\hat{C}_{t+k}\right) + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(\sum_{h=0}^{k}\hat{\Pi}_{t+h}\right) \\
&= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(M\hat{C}_{t+k}\right) + \sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(\hat{\Pi}_{t+k}\right) \\
&= \theta\beta E_t\left(\hat{P}_{t+1}^* - \hat{P}_t\right) + (1-\theta\beta)M\hat{C}_t + \hat{\Pi}_t
\end{aligned}
\tag{0.47}
$$

We have used the following property to get from the 3rd to the 4th line of (0.47):

$$
\begin{aligned}
&(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k \sum_{h=0}^{k}\hat{\Pi}_{t+h} \\
&= (1-\theta\beta)\left[\hat{\Pi}_t + \theta\beta\hat{\Pi}_{t+1} + \theta\beta\hat{\Pi}_t + (\theta\beta)^2\hat{\Pi}_{t+2} + (\theta\beta)^2\hat{\Pi}_{t+1} + (\theta\beta)^2\hat{\Pi}_t + \ldots\right] \\
&= \hat{\Pi}_t + \theta\beta\hat{\Pi}_{t+1} + \theta\beta\hat{\Pi}_t + (\theta\beta)^2\hat{\Pi}_{t+2} + (\theta\beta)^2\hat{\Pi}_{t+1} + (\theta\beta)^2\hat{\Pi}_t + \ldots \\
&\quad -\theta\beta\hat{\Pi}_t - (\theta\beta)^2\hat{\Pi}_{t+1} - (\theta\beta)^2\hat{\Pi}_t - (\theta\beta)^3\hat{\Pi}_{t+2} - (\theta\beta)^3\hat{\Pi}_{t+1} - (\theta\beta)^3\hat{\Pi}_t - \ldots \\
&= \sum_{k=0}^{\infty}(\theta\beta)^k\hat{\Pi}_{t+k}
\end{aligned}
$$

(We have dropped the expectations operator for notational ease.) To see why the second last line of (0.47) is the same as the last line

$$
\begin{aligned}
\hat{P}_t^* - \hat{P}_{t-1} &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(M\hat{C}_{t+k}\right) + \sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(\hat{\Pi}_{t+k}\right) \\
&= (1-\theta\beta)M\hat{C}_t + (1-\theta\beta)\sum_{k=1}^{\infty}(\theta\beta)^k E_t\left(M\hat{C}_{t+k}\right) + \hat{\Pi}_t + \sum_{k=1}^{\infty}(\theta\beta)^k E_t\left(\hat{\Pi}_{t+k}\right) \\
&= (1-\theta\beta)M\hat{C}_t + \hat{\Pi}_t + \theta\beta E_t\left[(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_{t+1}\left(M\hat{C}_{t+1+k}\right) + \sum_{k=0}^{\infty}(\theta\beta)^k E_{t+1}\left(\hat{\Pi}_{t+1+k}\right)\right] \\
&= (1-\theta\beta)M\hat{C}_t + \hat{\Pi}_t + \theta\beta E_t\left(\hat{P}_{t+1}^* - \hat{P}_t\right)
\end{aligned}
$$

The second last line of (0.47) tells us that a firm that is allowed to change its price wishes to deviate from last period's aggregate price level such that it keeps up with the expected future aggregate price level during the life of the price (second term) and such that any gap between expected and desired markup can be avoided (first term).

The aggregate price index (0.35) can be rewritten as

$$
P_t^{1-\varepsilon} = (1-\theta)P_t^{*1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \tag{0.48}
$$

We linearize by taking the first order taylor approximation around the steady-state

$$
\begin{aligned}
(1-\varepsilon)P^{-\varepsilon}\left(P_t - P\right) &= (1-\theta)(1-\varepsilon)P^{*-\varepsilon}\left(P_t^* - P^*\right) + \theta(1-\varepsilon)P^{-\varepsilon}\left(P_{t-1} - P\right) \\
P^{1-\varepsilon}\frac{P_t - P}{P} &= P^{*1-\varepsilon}(1-\theta)\frac{P_t^* - P^*}{P^*} + \theta P^{1-\varepsilon}\frac{P_{t-1} - P}{P} \\
\hat{P}_t &= (1-\theta)\hat{P}_t^* + \theta\hat{P}_{t-1}
\end{aligned}
\tag{0.49}
$$

Substract $\hat{P}_{t-1}$ from (0.49) to get

$$\hat{P}_t - \hat{P}_{t-1} = (1-\theta)\hat{P}_t^* + (\theta - 1)\hat{P}_{t-1}$$
$$\hat{\Pi}_t = (1-\theta)\left(\hat{P}_t^* - \hat{P}_{t-1}\right) \quad (0.50)$$

and substitute in (0.47) to finally get the New Phillips curve

$$\hat{P}_t^* - \hat{P}_{t-1} = \theta\beta E_t\left(\hat{P}_{t+1}^* - \hat{P}_t\right) + (1-\theta\beta)M\hat{C}_t + \hat{\Pi}_t$$
$$\hat{\Pi}_t(1-\theta)^{-1} = \theta\beta E_t\left(\hat{\Pi}_{t+1}(1-\theta)^{-1}\right) + (1-\theta\beta)M\hat{C}_t + \hat{\Pi}_t \quad (0.51)$$
$$\hat{\Pi}_t = \beta E_t\left(\hat{\Pi}_{t+1}\right) + \lambda M\hat{C}_t$$

where $\lambda = (1-\theta)(1-\theta\beta)/\theta$.

3. Summary of results

We now have derived all necessary (mostly non-linear) equations to perform equilibrium analysis. We will start by recalling the essential equations.

Household optimization
Labour supply

$$\frac{W_t}{P_t} = \frac{N_t^{\varphi}}{C_t^{-\sigma}} \tag{0.52}$$

Euler equation for consumption

$$C_t^{-\sigma} = \beta(1+i_t)E_t C_{t+1}^{-\sigma}\frac{P_t}{P_{t+1}} \tag{0.53}$$

Money demand

$$\frac{M_t}{P_t} = C_t^{\frac{\sigma}{\nu}}\left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{\nu}} \tag{0.54}$$

Firms optimization
Labour demand

$$\frac{W_t}{P_t} = A_t MC_t \tag{0.55}$$

New Phillips curve

$$\hat{\Pi}_t = \beta E_t\left(\hat{\Pi}_{t+1}\right) + \lambda M\hat{C}_t \tag{0.56}$$

where $\lambda = (1-\theta)(1-\theta\beta)/\theta$.

4. Log-linearization

To facilitate equilibrium analysis, we will continue by log-linearizing the above recalled equations. Variables with circumflex will denote %-deviation from steady-state. Lower case letters (without circumflex) denote logs of the level of the original variable.

Log-linearizing labour supply is straight forward since it is linear in logs. We get

$$\hat{w}_t - \hat{p}_t = \varphi \hat{n}_t + \sigma \hat{c}_t \tag{0.57}$$

The Euler equation for consumption is linearized around the steady-state as follows

$$\begin{aligned}
C_t^{-\sigma} &= \beta\left(1+i_t\right) E_t C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \\
C_t^{-\sigma} &= \beta R_t E_t C_{t+1}^{-\sigma} \frac{1}{\Pi_{t+1}} \\
-\sigma C^{-\sigma-1}\left(C_t - C\right) &= \beta C^{-\sigma} \frac{1}{\Pi}\left(R_t - R\right) - \sigma\beta R C^{-\sigma-1} \frac{1}{\Pi} E_t\left(C_{t+1} - C\right) \\
&\quad -\beta R C^{-\sigma} \frac{1}{\Pi^2} E_t\left(\Pi_{t+1} - \Pi\right) \\
-\sigma C^{-\sigma} \frac{C_t - C}{C} &= \beta R C^{-\sigma-1} \frac{1}{\Pi}\left[\frac{R_t - R}{R} - \sigma E_t\left(\frac{C_{t+1} - C}{C}\right) - E_t\left(\frac{\Pi - \Pi_{t+1}}{\Pi}\right)\right] \\
-\sigma \hat{c}_t &= \hat{R}_t - \sigma E_t \hat{c}_{t+1} - E_t \hat{\Pi}_{t+1} \\
\hat{c}_t &= -\frac{1}{\sigma}\left(\hat{R}_t - E_t \hat{\Pi}_{t+1}\right) + E_t \hat{c}_{t+1}
\end{aligned} \tag{0.58}$$

Linearizing the money demand is a bit trickier, and we show its linearization in appendix A just for the sake of completeness.

The labour demand curve is linear in logs so we get

$$\hat{w}_t - \hat{p}_t = \hat{a}_t + M\hat{C}_t \tag{0.59}$$

The New Phillips curve is already log-linearized.

$$\hat{\Pi}_t = \beta E_t\left(\hat{\Pi}_{t+1}\right) + \lambda M\hat{C}_t \tag{0.60}$$

5. Going from log-linearized to logs-of-levels

5.1 New "IS" curve

For monetary policy analysis it is nicer to have the equations expressed in logs of levels instead of %-deviations from steady-state. Some rewriting will do for the Euler equation. By the market clearing condition we substitute output for consumption and we have

$$\hat{y}_t = -\frac{1}{\sigma}\left(\hat{R}_t - E_t\hat{\Pi}_{t+1}\right) + E_t\hat{y}_{t+1}$$

$$\ln\frac{Y_t}{Y} = -\frac{1}{\sigma}\left(\ln\left(\frac{1+i_t}{1+i}\right) - E_t\ln\left(\frac{\Pi_{t+1}}{\Pi}\right)\right) + E_t\ln\frac{Y_{t+1}}{Y}$$

$$y_t = -\frac{1}{\sigma}\left(\ln(1+i_t) - \ln(1+i) - E_t\ln(\Pi_{t+1}) + \ln(\Pi)\right) + E_t y_{t+1} \qquad (0.61)$$

$$y_t \approx -\frac{1}{\sigma}\left(i_t + \ln(\beta) - E_t\ln(1+\pi_{t+1})\right) + E_t y_{t+1}$$

$$y_t = -\frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) + E_t y_{t+1}$$

where $y_t = \log Y_t$, i_t is the nominal interest rate, π_{t+1} is the inflation rate, $-\ln(\beta) = \rho$ and we used the steady-state implication from the Euler equation

$$C^{-\sigma} = \beta(1+i)C^{-\sigma}\frac{1}{\Pi}$$

$$(1+i) = \frac{\Pi}{\beta}$$

to substitute for $\ln(1+i)$.

5.2 New Phillips curve

To obtain an expression in logs-of-levels for the new Phillips curve, we will first introduce the output gap. We call the flexible prices equilibrium the second best outcome. Remember that even if we do not have price rigidity, we still have sub-optimality coming from monopolistic competition, i.e. firms pricing at a markup over marginal cost, which is why we call it only second best. Let us define the output gap $\hat{x}_t$ as the deviation of the staggered-pricing (log-linearized) output $\hat{y}_t$ from the flexible price (log-linearized) output $\hat{\bar{y}}_t$ and note that by definition

$$\hat{x}_t = \hat{y}_t - \hat{\bar{y}}_t = \ln\left(\frac{Y_t}{Y}\right) - \ln\left(\frac{\bar{Y}_t}{Y}\right) = y_t - \bar{y}_t = x_t \tag{0.62}$$

In order to express the %-deviation of marginal cost from its steady-state value in terms of the output gap, we proceed as follows:
The log-linearized market clearing condition is given by

$$\hat{y}_t = \hat{a}_t + \hat{n}_t = \hat{c}_t \tag{0.63}$$

Combining (0.63), (0.57) and (0.59) we get

$$\begin{aligned} \hat{w}_t - \hat{p}_t &= \hat{a}_t + M\hat{C}_t = \varphi \hat{n}_t + \sigma \hat{c}_t \\ M\hat{C}_t &= \varphi\left(\hat{y}_t - \hat{a}_t\right) + \sigma \hat{y}_t - \hat{a}_t \\ M\hat{C}_t &= \left(\varphi + \sigma\right)\hat{y}_t - \left(\varphi + 1\right)\hat{a}_t \end{aligned} \tag{0.64}$$

In the flex-price equilibrium the ex-post markup does not deviate from desired markup (marginal cost is constant when prices are flexible), so that we have

$$\begin{aligned} 0 &= \left(\varphi + \sigma\right)\hat{\bar{y}}_t - \left(\varphi + 1\right)\hat{a}_t \\ \hat{\bar{y}}_t &= \varphi_a \hat{a}_t \end{aligned} \tag{0.65}$$

where $\hat{\bar{y}}_t$ denotes the %-deviation from steady-state of the flexible prices output and $\varphi_a = \left(\varphi + 1\right)/\left(\varphi + \sigma\right)$. Substituting back in (0.64) we get

$$\begin{aligned} M\hat{C}_t &= \left(\varphi + \sigma\right)\left(\hat{y}_t - \hat{\bar{y}}_t\right) \\ &= \left(\varphi + \sigma\right)\hat{x}_t \end{aligned} \tag{0.66}$$

The New Phillips curve in terms of the output gap is

$$\hat{\Pi}_t = \beta E_t\left(\hat{\Pi}_{t+1}\right) + \kappa \hat{x}_t \tag{0.67}$$

where $\kappa = \left(\varphi + \sigma\right)\lambda > 0$. Which is by definition the same as

$$\hat{\Pi}_t = \beta E_t\left(\hat{\Pi}_{t+1}\right) + \kappa x_t \tag{0.68}$$

If we further make the assumption that we have zero inflation at the steady-state, we have

$$\hat{\Pi}_t = \ln\left(\frac{\Pi_t}{\Pi}\right) = \ln\left(1 + \pi_t\right) - \ln\left(1 + \pi\right) = \ln\left(1 + \pi_t\right) \approx \pi_t$$

which implies for the New Phillips curve

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa x_t \tag{0.69}$$

By rewriting it in terms of the ouptut gap and by making the additional zero inflation assumption we obtained an expression in function of the inflation rate and logs of levels of output gap.

[Note: We could have obtained a pricing rule in logs-of-levels from (0.46) had we introduced the zero inflation assumption earlier:

$$\hat{\Pi}^*_{t-1,t} = \ln\left(\frac{\Pi^*_{t-1,t}}{\Pi^*}\right) = \ln\left(\Pi^*_{t-1,t}\right) - \ln(1) = \ln\left(\frac{P^*_t}{P_{t-1}}\right) = p^*_t - p_{t-1}$$

$$\hat{\Pi}_{t-1,t+k} = \ln\left(\frac{\Pi_{t-1,t+k}}{\Pi}\right) = \ln\left(\Pi_{t-1,t+k}\right) - \ln(1) = \ln\left(\frac{P_{t+k}}{P_{t-1}}\right) = p_{t+k} - p_{t-1}$$

$$M\hat{C}_{t+k} = \ln\left(\frac{MC^n_{t+k} / P_{t+k}}{MC^n / P}\right) = \ln\left(\frac{MC^n_{t+k}}{P_{t+k}}\right) - \ln\left(\frac{MC^n}{P}\right) = mc^n_{t+k} - p_{t+k} - mc$$

so we can rewrite (0.46) to get the *linear pricing rule* in logs

$$\begin{aligned} p^*_t - p_{t-1} &= (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(mc^n_{t+k} - p_{t+k} - mc + p_{t+k} - p_{t-1}\right) \\ p^*_t &= \mu + (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(mc^n_{t+k}\right) \end{aligned} \quad (0.70)$$

where $\mu = -mc$.]

6. Flexible prices equilibrium

An equivalent expression to (0.65) can be obtained in logs of levels only. Just take logs of the original market clearing condition, labour supply and labour demand equations (instead of log-linearizing as above) and we will get an expression for the log of flex-prices *equilibrium output*

$$\begin{aligned} MC_t &= (\varphi+\sigma)\, y_t - (\varphi+1)\, a_t \\ -\mu &= (\varphi+\sigma)\, \overline{y}_t - (\varphi+1)\, a_t \\ \overline{y}_t &= \varphi_a a_t - \frac{\mu}{(\varphi+\sigma)} \end{aligned} \tag{0.71}$$

by recalling from (0.32) that marginal cost is constant ($MC_t = -\mu$) when prices are flexible. Using the market clearing condition we get an expression for the *equilibrium employment*

$$\overline{n}_t = (\varphi_a - 1)\, a_t - \frac{\mu}{(\varphi+\sigma)} \tag{0.72}$$

To get an expression for the *natural real rate* of interest $\overline{rr}_t$, recall (0.61) and use (0.71)

$$\begin{aligned} \overline{y}_t &= -\frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) + E_t\overline{y}_{t+1} \\ \varphi_a a_t - \frac{\mu}{(\varphi+\sigma)} &= -\frac{1}{\sigma}\left(\overline{rr}_t - \rho\right) + E_t\left(\varphi_a a_{t+1} - \frac{\mu}{(\varphi+\sigma)}\right) \\ \frac{1}{\sigma}\left(\overline{rr}_t - \rho\right) &= \varphi_a E_t\left(a_{t+1} - a_t\right) \\ \overline{rr}_t &= \sigma\varphi_a E_t\left(\Delta a_{t+1}\right) + \rho \\ \overline{rr}_t &= \sigma\varphi_a \rho_a \Delta a_t + \rho \end{aligned} \tag{0.73}$$

7. Equilibrium with price rigidities

We have already derived the **New Phillips curve**

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa x_t \tag{0.74}$$

Note that we can solve forward (0.74) if β is close to but smaller than one

$$\pi_t = \kappa \sum_{k=0}^{\infty} \beta^k E_t\left(x_{t+k}\right)$$

We see that current inflation is determined in a completely forward-looking manner: Only future expected deviations of output from second best (flex-prices) output matter for current inflation.

The **new „IS" curve** can be rewritten in terms of the output gap and the natural real rate

$$\begin{aligned} y_t - \overline{y}_t &= -\frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) + E_t y_{t+1} - E_t\overline{y}_{t+1} - \overline{y}_t + E_t\overline{y}_{t+1} \\ x_t &= -\frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) + E_t x_{t+1} + E_t\Delta\overline{y}_{t+1} \end{aligned} \tag{0.75}$$

Recall from above that the natural rate of interest $\overline{rr}_t$ is the real rate of interest that prevails in the flexible prices setup (set $x_t = 0$ in (0.75) or c.f. flexible prices equilibrium)

$$\overline{rr}_t = \rho + \sigma E_t \Delta\overline{y}_{t+1} \tag{0.76}$$

and since we assumed log-techonlogy a_t to follow an exogenous difference-stationary stochastic process we have

$$\overline{rr}_t = \rho + \sigma\varphi_a E_t \Delta a_{t+1} \tag{0.77}$$

The new „IS" curve can finally be written in terms of the natural rate of interest

$$\begin{aligned} x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \overline{rr}_t + \sigma\varphi_a E_t\Delta a_{t+1}\right) + \varphi_a E_t \Delta a_{t+1} \\ &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \overline{rr}_t\right) \end{aligned} \tag{0.78}$$

Today's output gap is thus determined by the expected future output gap and the gap between the expected real interest rate and the natural interest rate.
Note that since $\sigma^{-1} < 1$, (0.78) can be iterated forward to get

$$x_t = -\frac{1}{\sigma}\sum_{k=0}^{\infty} E_t\left(i_{t+k} - E_t\pi_{t+1+k} - \overline{rr}_{t+k}\right) \tag{0.79}$$

The current output gap is thus only affected by expected future deviations of the expected real rate $\left(i_{t+k} - E_t\pi_{t+1+k}\right)$ from the natural rate of interest $\overline{rr}_{t+k}$.

The money demand, equations (0.74) and (0.78) describe a closed system that we can use for equilibrium and monetary policy analysis. However, the assumption that the nominal interest rate serves as the instrument of monetary policy makes the money demand equation redundant because the central bank is supposed to immediately adjust money supply such that the interest rate is where it wants it to be.

8. Optimal monetary policy

In this section we will discuss the implications of the results of New Neoclassical Synthesis for optimal monetary policy. In a first part we will suppose that the central bank does not commit and we will see how it should behave in the time consistent framework. In the second part we will assume the central bank uses a commitment technology and we will see its implications. We will derive the results stated in Clarida Gali Gerlter 1999 (CGG99 hereinafter).

8.1 The setup

The central bank has the objective function

$$-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}-k\right)^2+\left(\pi_{t+i}-l\right)^2\right)\right] \tag{0.80}$$

where β is the time preference discount factor, x_t the output gap and π_t the inflation rate. The parameter α indicates the relative weight put on the deviation of current output from long run output (=output gap). The parameters k and l denote the target value for the output gap and inflation respectively, both of which we will assume to be zero in the first part.
The central bank is faced with the task to maximize (0.80) subject to the Phillips curve (a.k.a. AS curve)

$$\begin{aligned}\pi_t &= \beta E_t\left(\pi_{t+1}\right)+\kappa x_t+u_t \\ &= E_t\sum_{k=0}^{\infty}\beta^k\left(\kappa x_{t+k}+u_{t+k}\right)\end{aligned} \tag{0.81}$$

and subject to the IS curve

$$\begin{aligned}x_t &= E_t x_{t+1}-\frac{1}{\sigma}\left(i_t-E_t\pi_{t+1}-\overline{rr}_t\right) \\ &= -\frac{1}{\sigma}\sum_{k=0}^{\infty}E_t\left(i_{t+k}-E_t\pi_{t+1+k}-\overline{rr}_{t+k}\right)\end{aligned} \tag{0.82}$$

Note that we have introduced a cost push shock u_t in the Phillips curve. It is assumed to follow the AR(1) process $u_t=\rho u_{t-1}+\upsilon_t$ where $\upsilon_t \sim i.i.N\left(0,\sigma_\upsilon^2\right)$. This shock u_t represents (exogenous) factors – other than changes in output – that cause the real marginal cost to vary. We know that excess demand leads firms to produce more which increases their real marginal cost (cf. equation (0.66)) and now by introducing this cost push shock we allow marginal cost to be influenced by other factors than excess demand.
The optimal solution – i.e. the optimal feedback rule consists of a time path for the policy instrument i_t, which determines optimal time paths for x_t and π_t. Compared to the classical Tinbergein-Theil problem, the behaviour of the target variables now depends on expected future policy as well, as the second lines in (0.81) and (0.82) make clear.

8.2 Optimal monetary policy under discretion

We assume that the central bank reoptimizes the problem described in 8.1 every period. The feedback rule will thus relate the policy instrument to the current state of the economy. Due to rational expectations, the private sector takes this behaviour into account for the formation of expectations and in the equilibrium the central bank has no incentives to change its plans in an unexpected way, even though it has the discretion to do so. The solution is therefore said to be time consistent. The central bank is thus unable to influence future expectations and takes them as given in its optimization problem.
A convenient way to solve the optimization problem is to set up the Lagrangien as follows:

$$L = -\frac{1}{2} E_t \left[\sum_{i=0}^{\infty} \beta^i \left(\alpha x_{t+i}^2 + \pi_{t+i}^2 \right) \right] + \lambda \left(\pi_t - \beta E_t \left(\pi_{t+1} \right) - \kappa x_t - u_t \right) \tag{0.83}$$

The FOCs are

$$\begin{aligned} \frac{\partial L}{\partial x_t} &= -\alpha x_t + \lambda \kappa = 0 \\ \frac{\partial L}{\partial \pi_t} &= -\pi_t + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= \pi_t - \beta E_t \left(\pi_{t+1} \right) - \kappa x_t - u_t = 0 \end{aligned} \tag{0.84}$$

From the first two FOCs we get

$$x_t = -\frac{\kappa}{\alpha} \pi_t \tag{0.85}$$

The central bank should induce a negative output gap as soon as inflation is above target and vice versa. The intensity of the contraction depends negatively on the weight (α) attributed in the objective function to output stabilization and positively on the sensitivity of inflation with respect to the output gap (κ).
We can get reduced forms by substituting (0.85) into the Phillips curve. The reduced form expression for π_t is found as

$$\begin{aligned} \pi_t &= \beta E_t \left(\pi_{t+1} \right) + \kappa x_t + u_t \\ &= -\frac{\kappa^2}{\alpha} \pi_t + \beta E_t \left(\pi_{t+1} \right) + u_t \\ &= \frac{\alpha}{\alpha + \kappa^2} \left[\beta E_t \left(\pi_{t+1} \right) + u_t \right] \\ \pi_t &= \frac{\alpha}{\alpha + \kappa^2} E_t \sum_{k=1}^{\infty} \left(\frac{\beta \alpha}{\alpha + \kappa^2} \right)^k u_{t+k} \end{aligned} \tag{0.86}$$

Note that $E_t \left(u_{t+k} \right) = \rho^k u_t$, so we get

$$\begin{aligned} \pi_t &= \frac{\alpha}{\alpha + \kappa^2} u_t \sum_{k=1}^{\infty} \left(\frac{\beta \alpha \rho}{\alpha + \kappa^2} \right)^k \\ &= \frac{\alpha}{\alpha + \kappa^2} \frac{1}{1 - \frac{\beta \alpha \rho}{\alpha + \kappa^2}} u_t \\ &= \alpha \frac{1}{\kappa^2 + \alpha \left(1 - \beta \rho \right)} u_t \\ \pi_t &= \alpha q u_t \end{aligned} \tag{0.87}$$

The reduced form expression for x_t is found by substituting (0.87) into (0.85)

$$x_t = -\kappa q u_t \tag{0.88}$$

where $q = \dfrac{1}{\kappa^2 + \alpha(1-\beta\rho)}$.

We can now analyze how the choice of the parameter α influences the output gap and inflation variability:

$$\begin{aligned} \sigma_\pi &= \frac{\alpha}{\kappa^2 + \alpha(1-\beta\rho)}\sigma_u \\ \sigma_x &= \frac{\kappa}{\kappa^2 + \alpha(1-\beta\rho)}\sigma_u \end{aligned} \tag{0.89}$$

If $\alpha \to 0$, we have no inflation variability and a standard deviation of $\kappa^{-1}\sigma_u$ for the output gap. If $\alpha \to \infty$, we have no output gap variability and a standard deviation of $(1-\beta\rho)^{-1}\sigma_u$ for inflation. [cf. result 1 in CGG99]

If there are no cost push shocks, there is no trade-off between inflation and output gap variability! In the absence of cost push shocks, inflation targeting is thus the optimal monetary policy.

Using the IS curve, the we can finally find the optimal feedback rule

$$\begin{aligned} x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \overline{rr}_t\right) \\ \sigma\kappa q u_t &= \sigma\rho\kappa q u_t + i_t - \rho\alpha q u_t - \overline{rr}_t \\ i_t &= (\sigma\kappa q + \rho\alpha q - \sigma\rho\kappa q)u_t + \overline{rr}_t \\ i_t &= \left(1 + \frac{\sigma\kappa(1-\rho)}{\rho\alpha}\right)\rho\alpha q u_t + \overline{rr}_t \\ i_t &= \gamma_\pi \rho\alpha q u_t + \overline{rr}_t \end{aligned} \tag{0.90}$$

where $\gamma_\pi = 1 + \dfrac{\sigma\kappa(1-\rho)}{\rho\alpha} > 1$. Equivalently, using $E_t\pi_{t+1} = \rho\alpha q u_t$, we get

$$i_t = \gamma_\pi E_t \pi_{t+1} + \overline{rr}_t \tag{0.91}$$

From this optimal feedback rule we see that the central bank should increase the interest rate by more than the increase in expected inflation (cf. result 3 in CGG99). The central bank need not react to a shock that affects the long run output.

8.3 Optimal monetary policy with commitment

If we assume commitment, the problem of time inconsistency arises. Once the central bank commits and the economic agents have formed their expectations accordingly, the central bank has incentives to deviate from its commitment. The issue of central bank credibility becomes crucial because rational agents will not let the central bank repeatedly fool them. If the central bank tries for example to keep the output gap at positive values, the problem of inflationary bias arises.

We will first treat the case with inflationary bias through altering the setup by no longer keeping k at zero but rather at a positive value $k>0$ in the objective function of the central bank in order to specify a positive target value for the output gap.

$$-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}-k\right)^2+\pi_{t+i}^2\right)\right] \tag{0.92}$$

The central bank seeks to maximize (0.92) subject to (0.82) and to a Phillips curve

$$\pi_t=E_t\left(\pi_{t+1}\right)+\kappa x_t+u_t \tag{0.93}$$

where we have set $\beta=1$ in order to simplify calculations.
Proceeding the same way as above, we form the Lagrangeien

$$L=-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}-k\right)^2+\pi_{t+i}^2\right)\right]+\lambda\left(\pi_t-E_t\left(\pi_{t+1}\right)-\kappa x_t-u_t\right) \tag{0.94}$$

The FOCs are

$$\begin{aligned}
&L=-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}-k\right)^2+\pi_{t+i}^2\right)\right]+\lambda\left(\pi_t-E_t\left(\pi_{t+1}\right)-\kappa x_t-u_t\right)\\
&\frac{\partial L}{\partial x_t}=\alpha\left(x_{t+i}-k\right)-\lambda\kappa=0\\
&\frac{\partial L}{\partial \pi_t}=\pi_t+\lambda=0\\
&\frac{\partial L}{\partial \lambda}=\pi_t-E_t\left(\pi_{t+1}\right)-\kappa x_t-u_t=0
\end{aligned} \tag{0.95}$$

We introduce notation π_t^k and x_t^k to explicitly denote solutions with positive output gap target k. From the first two we get the optimality condition

$$x_t^k=-\frac{\kappa}{\alpha}\pi_t^k+k \tag{0.96}$$

Substituting in (0.93) gives

$$\begin{aligned}
\pi_t^k&=E_t\left(\pi_{t+1}^k\right)-\frac{\kappa^2}{\alpha}\pi_t^k+\kappa k+u_t\\
\pi_t^k&=\frac{\alpha}{\kappa^2+\alpha}\left(E_t\left(\pi_{t+1}^k\right)+\kappa k+u_t\right)\\
\pi_t^k&=\frac{\alpha}{\kappa^2+\alpha}E_t\sum_{j=0}^{\infty}\left(\frac{\alpha}{\kappa^2+\alpha}\right)^j\left(\kappa k+u_{t+j}\right)\\
\pi_t^k&=\frac{\alpha}{\kappa^2+\alpha}\left(\frac{\kappa}{1-\frac{\alpha}{\kappa^2+\alpha}}k+\frac{1}{1-\frac{\rho\alpha}{\kappa^2+\alpha}}u_t\right)\\
\pi_t^k&=\frac{\alpha}{\kappa}k+\frac{\alpha}{\kappa^2+\alpha\left(1-\rho\right)}u_t
\end{aligned} \tag{0.97}$$

By (0.87) we can relate π_t^k to π_t

$$\pi_t^k=\alpha q u_t+\frac{\alpha}{\kappa}k=\pi_t+\frac{\alpha}{\kappa}k \tag{0.98}$$

and using this last expression and (0.85), we have for the output gap

$$x_t^k = -\frac{\kappa}{\alpha}\left(\pi_t + \frac{\alpha}{\kappa}k\right) + k = -\frac{\kappa}{\alpha}\pi_t = x_t \qquad (0.99)$$

From expression (0.98) we see how the greater-than-zero k output gap target causes an inflationary bias, without resulting in a greater (positive) output gap (0.99). (cf. result 5 in CGG99). One popular solution that has been suggested to counteract the inflationary bias problem is to appoint a conservative central bank chairman who assigns a higher relative cost to inflation than society as a whole, i.e. with a $\alpha^R < \alpha$. (cf. result 6 in CGG99).

We now look at the case where the central bank can credibly commit and see that this leads to an even better outcome than discretion. For the sake of simplicity we will first analyse the case where the central bank adopts a rule similar to the discretion result (0.88). Consider

$$x_t^c = -\omega u_t \qquad (0.100)$$

where the superscript c explicitly denotes the output gap under (credible) commitment and ω is the parameter to be determined by the central bank through optimization. First, note that the optimal feedback rule for inflation is

$$\begin{aligned}\pi_t^c &= E_t \sum_{k=0}^{\infty} \beta^k \left(\kappa x_{t+k}^c + u_{t+k}\right) \\ &= \sum_{k=0}^{\infty} \beta^k \left(-\omega\kappa u_{t+k} + E_t u_{t+k}\right) \\ &= \sum_{k=0}^{\infty} \beta^k \left(1-\omega\kappa\right) E_t u_{t+k} \\ &= \left(1-\omega\kappa\right) u_t \sum_{k=0}^{\infty} \left(\beta\rho\right)^k \\ &= \frac{1-\omega\kappa}{1-\beta\rho} u_t\end{aligned} \qquad (0.101)$$

and substituting back in the policy rule

$$\begin{aligned}\pi_t^c &= \frac{1-\omega\kappa}{1-\beta\rho} u_t = \frac{-\omega\kappa}{1-\beta\rho} u_t + \frac{1}{1-\beta\rho} u_t \\ &= \frac{\kappa}{1-\beta\rho} x_t^c + \frac{1}{1-\beta\rho} u_t\end{aligned} \qquad (0.102)$$

Already now we can see an improvement due to commitment. Since the central bank can influence the future expectations, it faces an improved short run output gap vs. inflation trade-off. We have $\frac{\kappa}{1-\beta\rho} > \kappa$ which implies that a smaller variation in the output gap will cause the same impact on inflation as under discretion.

Using expressions (0.100) and (0.101) we can write the objective function for the central bank as follows

$$\begin{aligned}&\max_{\omega}\left\{-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(-\omega u_{t+i}\right)^2 + \left(\frac{1-\omega\kappa}{1-\beta\rho}u_{t+i}\right)^2\right)\right]\right\} \\ &\max_{\omega}\left\{-\frac{1}{2}\left[\alpha\left(\omega u_t\right)^2 + \left(\frac{1-\omega\kappa}{1-\beta\rho}u_t\right)^2\right]E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\frac{u_{t+i}}{u_t}\right)^2\right]\right\}\end{aligned} \qquad (0.103)$$

where we multiplied and divided by u_t^2. The FOC is

$$\frac{\partial}{\partial\omega}\left[\alpha\left(\omega u_t\right)^2+\left(\frac{1-\omega\kappa}{1-\beta\rho}u_t\right)^2\right]=0$$

$$2\alpha\omega u_t+2\frac{1-\omega\kappa}{1-\beta\rho}\frac{-\kappa}{1-\beta\rho}u_t=0$$

$$\alpha\left(1-\beta\rho\right)^2\omega+\omega\kappa^2=\kappa \tag{0.104}$$

$$\omega=\frac{\kappa}{\alpha\left(1-\beta\rho\right)^2+\kappa^2}$$

In order to get an expression similar to the discretion parameter q, let us write

$$\omega=\frac{\kappa}{\alpha\left(1-\beta\rho\right)^2+\kappa^2}=\frac{\kappa}{\kappa^2+\alpha^c\left(1-\beta\rho\right)} \tag{0.105}$$

where $\alpha^c=\alpha\left(1-\beta\rho\right)$. If we rewrite the second line in (0.104) we get the optimality condition

$$\alpha\omega u_t=\frac{1-\omega\kappa}{1-\beta\rho}\frac{\kappa}{1-\beta\rho}u_t$$

$$-\alpha x_t^c=\frac{\kappa}{1-\beta\rho}\pi_t^c \tag{0.106}$$

$$x_t^c=-\frac{\kappa}{\alpha^c}\pi_t^c$$

This condition leads the central bank to induce a larger negative output gap in reaction to positive inflation than in the discretion case since $\alpha^c<\alpha$. This will induce stronger interest rate reactions with respect to expected changes in inflation

$$i_t=\gamma_\pi^c E_t\pi_{t+1}+\overline{rr}_t \tag{0.107}$$

where $\gamma_\pi^c=1+\frac{\sigma\kappa\left(1-\rho\right)}{\rho\alpha^c}>\gamma_\pi>1$.

The reduced forms for the output gap and inflation are

$$\begin{aligned} x_t^c&=-\kappa q^c u_t\\ \pi_t^c&=\alpha^c q^c u_t \end{aligned} \tag{0.108}$$

where $q^c=\frac{1}{\kappa^2+\alpha^c\left(1-\beta\rho\right)}$.

We have seen that if a central bank can credibly commit, then it faces an improved short run trade-off between the output gap and inflation and because it can influence future expectations. The central bank should react stronger to deviations of inflation from target. (cf. result 7 in CGG99).

We should be aware that the above used optimality condition $x_t^c=-\omega u_t$ is not necessarily the global optimum. We now want to solve for a general solution, so let us set up the optimization problem

$$\max_{x_{t+i},\pi_{t+i}|_{i=0}^{\infty}}\left\{-\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}\right)^2+\left(\pi_{t+i}\right)^2\right)\right]\right\} \tag{0.109}$$

subject to

$$\pi_{t+i} = \beta E_t\left(\pi_{t+i+1}\right) + \kappa x_{t+i} + u_{t+i} \quad \text{for } i = 0,\ldots,\infty \tag{0.110}$$

The Lagrangien yields

$$L = -\frac{1}{2}E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\alpha\left(x_{t+i}\right)^2 + \left(\pi_{t+i}\right)^2 + \phi_{t+i}\left(\pi_{t+i} - \beta\pi_{t+i+1} - \kappa x_{t+i} - u_{t+i}\right)\right)\right] \tag{0.111}$$

where ϕ_{t+i} denotes the Lagrange multiplier of period $t+i$. The FOCs are

FOC1:

$$\frac{\partial L}{\partial x_t} = -\frac{1}{2}\left[\alpha x_t + \frac{1}{2}\phi_t\kappa\right] = 0$$

$$\Rightarrow x_t = -\frac{1}{2}\frac{\kappa}{\alpha}\phi_t$$

FOC2:

$$\frac{\partial L}{\partial \pi_t} = -\frac{1}{2}\left[2\pi_t + \phi_t\right] = 0$$

$$\Rightarrow \phi_t = 2\pi_t$$

FOC3:

$$\frac{\partial L}{\partial x_{t+i}} = -\frac{1}{2}E_t\left[2\beta^i\alpha x_{t+i} - \kappa\beta^i\phi_{t+i}\right]$$

$$\Rightarrow \phi_{t+i} = 2\frac{\alpha}{\kappa}x_{t+i}$$

FOC4:

$$\frac{\partial L}{\partial \pi_{t+i}} = -\frac{1}{2}E_t\left[2\beta^i\pi_{t+i} + \beta^i\phi_{t+i} - \beta^{i-1}\beta\phi_{t+i-1}\right] = 0$$

$$\Rightarrow \pi_{t+i} = -\frac{1}{2}\left[\phi_{t+i} - \phi_{t+i-1}\right]$$

Substituting FOC1 into FOC2 we get the rule for the current time t

$$x_t = -\frac{\kappa}{\alpha}\pi_t \tag{0.112}$$

and substituting FOC3 into FOC4 and rearranging the terms we get the rule for times $t+i$ where $i = 1,\ldots,\infty$

$$x_{t+i} - x_{t+i-1} = -\frac{\kappa}{\alpha}\pi_{t+i} \tag{0.113}$$

The new optimality rule for $t+i$ is a difference equation in the output gap. Substituting (0.113) twice into the Phillips curve we get a stochastic difference equation in x_t

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa x_t + u_t$$

$$-\frac{\alpha}{\kappa}\left(x_t - x_{t-1}\right) = -\frac{\alpha}{\kappa}\beta E_t\left(x_{t+1} - x_t\right) + \kappa x_t + u_t$$

$$x_t\left(\alpha + \alpha\beta + \kappa^2\right) = \alpha x_{t-1} + \alpha\beta E_t\left(x_{t+1}\right) - \kappa u_t \tag{0.114}$$

$$x_t = \frac{\alpha}{\alpha\left(1+\beta\right)+\kappa^2}x_{t-1} + \frac{\alpha}{\alpha\left(1+\beta\right)+\kappa^2}\beta E_t\left(x_{t+1}\right) - \frac{\alpha}{\alpha\left(1+\beta\right)+\kappa^2}\frac{\kappa}{\alpha}u_t$$

$$x_t = c x_{t-1} + c\beta E_t\left(x_{t+1}\right) - c\frac{\kappa}{\alpha}u_t$$

where $c=\frac{\alpha}{\alpha\left(1+\beta\right)+\kappa^2}$. We can solve for the stationary solution of x_t by the method of unobserved components. Assume the process is given by

$$\begin{aligned} x_t &= \delta x_{t-1} + e u_t \\ x_{t+1} &= \delta x_t + e u_{t+1} \end{aligned} \tag{0.115}$$

and substitute in (0.114)

$$\begin{aligned} x_t &= c x_{t-1} + c\beta E_t\left(x_{t+1}\right) - c\frac{\kappa}{\alpha}u_t \\ x_t &= c x_{t-1} + c\beta E_t\left(\delta x_t + e u_{t+1}\right) - c\frac{\kappa}{\alpha}u_t \\ x_t &= c x_{t-1} + c\beta E_t\left(\delta^2 x_{t-1} + \delta e u_t\right) + c\beta\rho e u_t - c\frac{\kappa}{\alpha}u_t \\ x_t &= x_{t-1}\left(c + c\beta\delta^2\right) + u_t\left(c\beta\delta e + c\beta\rho e - c\frac{\kappa}{\alpha}\right) \end{aligned} \tag{0.116}$$

hence we know that

$$\begin{aligned} \delta &= c + c\beta\delta^2 \\ 0 &= c\beta\delta^2 + \delta + c \\ \delta &= \frac{1 \pm \sqrt{1-4\beta c^2}}{2\beta c} = \frac{1-\sqrt{1-4\beta c^2}}{2\beta c} \end{aligned} \tag{0.117}$$

and

$$\begin{aligned} e &= c\beta\delta e + c\beta\rho e - c\frac{\kappa}{\alpha} \\ e &= \frac{-c\kappa}{\alpha\left(1-c\beta\delta-c\beta\rho\right)} \end{aligned} \tag{0.118}$$

so we get the stationary solution for the difference equation of the output gap

$$x_t = \delta x_{t-1} - \frac{c\kappa}{\alpha\left(1-c\beta\delta-c\beta\rho\right)}u_t \tag{0.119}$$

For the inflation process we get another difference equation.

$$\pi_t = \delta\pi_t + \frac{\delta}{1-\delta\beta\rho}\left(u_t - u_{t-1}\right) \tag{0.120}$$

First we note that this difference rule looks different than the discretion solution from which we can conclude that the discretion rule or the one derived just before (with a structure similar to the discretion rule) is not globally optimal.
Second, we see from expressions (0.119) and (0.120) that current policy depends on lagged variables x_{t-1} and u_{t-1}. CGG99 argue that this lagged dependence originates in the fact that the central bank can influence future expectations. Suppose there was a cost push shock causing $\pi_t > 0$. Under discretion we saw that the central bank contracts x_t and then lets x_{t+i} return back to trend as π_{t+i} returns to zero. Under commitment the central bank continues to reduce x_{t+i} until π_{t+i} is back on target. This credible threat of contracting x_{t+i} causes current inflation to jump up less in response to the original shock because it depends as well on future expected values of x_{t+i}.

We will just note that there might be a problem of indeterminacy in the resulting interest rate rule, without further elaborating on it.
(cf. result 8 in CGG99).

8.4 Further aspects of optimal monetary policy

Imperfect information: Result 9 in CGG99: “With imperfect information, stemming either from data problems or lags in the effect of policy, the optimal policy rules are the certainty equivalent versions of the perfect information case. Policy rules must be expressed in terms of the forecasts of target variables as opposed t the ex post behaviour. Using observable intermediate targets such as broad money aggregates is a possibility, but experience suggests that these indirect indicators are generally too unstable to be used in practice.”

Instrument choice: Result 10 in CGG99: “Large unobservable shocks to money demand produce high volatility of interest rates when a monetary aggregate is used as the policy instrument. It is largely for this reason that an interest rate instrument may be preferable.”

[left out: result 11: parameter uncertainty may lead to smoother path of interest rate.]
[left out: result 12: opportunistic disinflation à la Blinder is ok if more weight is attributed to small departures of output gap from target than small departures of inflation from target.]

8.5 Endogenous inflation and output gap persistence

We now introduce lagged terms of the output gap and inflation in the IS and Phillips curve

$$\begin{aligned} x_t &= \theta E_t x_{t+1} + (1-\theta) x_{t-1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \overline{rr}_t\right) \\ \pi_t &= \phi \beta E_t\left(\pi_{t+1}\right) + (1-\phi)\pi_{t-1} + \kappa x_t + u_t \end{aligned} \tag{0.121}$$

where θ and ϕ are weighting parameters for future and past values of output gap and inflation respectively. Suggested reasons of motivating the inclusion of lagged terms are some form of adjustment costs (IS curve) and costs of changing the rate of inflation or adaptive expectations (Phillips curve), but the true reason is simply that one empirically observes strong persistence. This modified Phillips curve is sometimes called hybrid Phillips curve.
The implications for optimal monetary policy derived before (results 1-4) remain virtually unchanged. The most notable aspect that changes is that monetary policy responds to future expected values of inflation since today’s policy actions now have an influence on future inflation as well due to inflation persistence.

Appendix A: Linearizing the money demand

In his paper Gali (2002) only "postulate[s], without deriving it, a standard money demand equation:"

$$m_t - p_t = \frac{\sigma}{\nu} c_t - \eta i_t \tag{0.122}$$

where the variables are logs of levels, except i_t which denotes the nominal interest rate. To get to this expression through deriving it from the consumer's optimization problem, we have to take a not-so-clear cut path: In the previous linearizations we clearly distinguished between log-linearizing and taking logs of levels, i.e. we did not mix the two linearization methods. But now, in order to linearize

$$\frac{M_t}{P_t} = C_t^{\frac{\sigma}{\nu}} \left(\frac{i_t}{1+i_t} \right)^{-\frac{1}{\nu}}$$

we first have to take logs and where there still is something non-linear, we make a first order Taylor approximation around the steady-state and ignore constant terms, i.e. we mix linearization procedures.

Taking logs and rewriting gives

$$\ln M_t - \ln P_t = \frac{\sigma}{\nu} \ln C_t - \frac{1}{\nu} \ln \left(1 - \frac{1}{1+i_t} \right)$$

$$m_t - p_t = \frac{\sigma}{\nu} c_t - \frac{1}{\nu} \ln \left(1 - \frac{1}{\exp\{\ln(1+i_t)\}} \right)$$

$$m_t - p_t = \frac{\sigma}{\nu} c_t - \frac{1}{\nu} \ln \left(1 - \frac{1}{\exp\{i_t\}} \right)$$

The second term on the RHS is non-linear, so we compute its Taylor approximation around the steady-state interest rate i

$$\ln \left(1 - \frac{1}{\exp\{i_t\}} \right) \approx \ln \left(1 - \frac{1}{\exp\{i\}} \right) + \frac{1}{1 - \frac{1}{\exp\{i\}}} \frac{1}{\exp\{i\}} (i_t - i)$$

$$\approx \ln \left(1 - \frac{1}{\exp\{i\}} \right) + \frac{\exp\{i\}}{\exp\{i\} - 1} \frac{1}{\exp\{i\}} (i_t - i)$$

$$\approx \ln \left(1 - \frac{1}{\exp\{i\}} \right) + \frac{1}{(1+i) - 1} (i_t - i)$$

$$\approx const + \frac{1}{i} i_t$$

Ignoring the constant terms, we can rewrite the money demand as

$$\begin{aligned} m_t - p_t &= \frac{\sigma}{\nu} c_t - \frac{1}{\nu} \frac{1}{i} i_t \\ m_t - p_t &= \frac{\sigma}{\nu} c_t - \eta i_t \end{aligned} \tag{0.123}$$

where η is in function of i.

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