Adaptive Learning

6.) Exercises

6.1) Simulation of the Lucas Model

Consider the Lucas aggregate supply model as given in the theory. Calibrate the model as follows:
The parameters take values: $\varphi = 5$, $\delta = 1$ and $\alpha = -0.5$.
The exogenous variable (we assume there is only one) follows a white noise process with $w_t \sim N(0,1)$. The disturbance term follows a white noise process with $\eta_t \sim N(0,0.5)$.

i) Does the model converge with the parameter values given in the setup?

Solution:
i) Yes, $\alpha < 1$.

ii) Compute the REE values of the coefficients $\bar{a}$ and $\bar{b}$.

Solution:
ii) From equation (0.14) we get

$$\frac{da}{d\tau} = \varphi + (\alpha - 1)a = 0 \Leftrightarrow a = \frac{-\varphi}{(\alpha - 1)} = \frac{10}{3}$$

$$\frac{db}{d\tau} = \delta + (\alpha - 1)b \Leftrightarrow b = \frac{-\delta}{(\alpha - 1)} = \frac{2}{3}$$

iii) Simulate the model! Use the starting values $a_0 = 1$ and $b_0 = 2$.
[i.e. equations

$$p_t = T(\boldsymbol{\varphi}_{t-1})' \mathbf{z}_t + \eta_t$$

$$\boldsymbol{\varphi}_t = \boldsymbol{\varphi}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{z}_{t-1}\left(\mathbf{z}'_{t-1}\left(T(\boldsymbol{\varphi}_{t-1}) - \boldsymbol{\varphi}_{t-1}\right) + \eta_t\right)$$

$$\mathbf{R}_t = \mathbf{R}_{t-1} + t^{-1}\left(z_{t-1}z'_{t-1} - \mathbf{R}_{t-1}\right)$$]

Solution:
iii)

```
delete *
close few

create few u 1 100

rndseed 123456789

scalar obs=100
scalar fi=5
scalar alpha=-0.5
scalar delta=1
```

```
vector(obs) w
nrnd(w)
vector(obs) n
nrnd(n)
n=n*0.5
vector(obs) p

matrix(2,2) R
R=@identity(2)

vector(2) z
z(1)=1
matrix(obs,2) phi
phi(1,1)=1
phi(1,2)=2

vector(2) tmap
vector(2) temp

for !i=2 to obs
	z(2)=w(!i-1)
	R=R+(z*@transpose(z)-R)/(!i)
	tmap(1)=(fi+alpha*phi(!i-1,1))
	tmap(2)=(delta+alpha*phi(!i-1,2))
	rowplace(phi,@transpose((@transpose(@rowextract(phi,!i-
1))+1/(!i)*@inverse(R)*z*(@transpose(z)*(tmap-@transpose(@rowextract(phi,!i-1)))+n(!i)))),!i)
	p(!i)=(@transpose(tmap)*z)(1)+n(!i)
next

series a
series b
group estimates a b
mtos(phi,estimates)
```

iv) Do your simulations converge? If no: re-do question iii)! If yes: how long does it take to converge to the REE? Describe (and explain) the behaviour of p_t and $\boldsymbol{\varphi}_t$.

Solution:
iv) Yes, it converges. The coefficient estimates take about 20 observations to converge. The price level is always fluctuating (because of the shocks η_t) around the level of 3.3.

v) What is your answer to question iv) when we have $\eta_t \sim N(0,2)$?

Solution:
v) It now takes far longer to converge.

vi) And if we had $w_t \sim N(0,4)$?

Solution:
vi) Normal speed of convergence!

6.2 Decreasing gains vs. constant gains

(This is just doing the model in the book Evans Honkapohja 2001 Chapter 3.3 Learning with Constant Gain)

In the theory we have seen that one of the necessary assumptions for convergence was "conditions on the rate at which $\gamma_t \to 0$". In this exercise we will see what happens if the gain sequence is constant $\gamma_t = \gamma$.

Consider the simple model

$$p_t = \alpha + \beta p_{t+1}^e + v_t$$

where p_t^e denotes expected prices (based on information up to time t) and v_t is a white noise disturbance process.

i) Find the REE of this model! (Note that in this case, the REE is the stochastic steady-state.)

Solution:

$$p_t = \alpha + \beta p_{t+1}^e + v_t$$

$$E[p_t] = \alpha + \beta E[p_{t+1}]$$

at s.s. we have $p_t = p_{t+1}$

$$E[p_t] = \frac{\alpha}{1-\beta}$$

rational expectations: $p_t - E[p_t] = \varepsilon_t$

$$p_t = \frac{\alpha}{1-\beta} + \varepsilon_t = \overline{a} + \varepsilon_t$$

ii) Assuming that the PLM takes the form of the REE, find the ALM.

Solution:

PLM :

$$p_t = a + v_t \Rightarrow p_t^e = a$$

ALM :

$$p_t = \alpha + \beta p_{t+1}^e + v_t$$

$$p_t = \alpha + \beta a + v_t$$

iii) Using the T-map, determine the conditions (on the parameter values) for E-stability!

Solution:

$$T(a) = \alpha + \beta a$$

$$\begin{aligned}\frac{d}{d\tau}(a) &= T(a) - a \\ &= \alpha + \beta a - a \\ &= \alpha + (\beta - 1)a\end{aligned}$$

We need that $\beta < 1$.

iv) Now, assume that the condition found in iii) is satisfied. Find the SRA for a_t and determine the convergence properties via the associated ODE. (Hint: Recall the formula for updating an estimated mean derived in the exercises of the recursive least squares!)

Solution:

In ii) we saw that $p_t^e = a$, hence a is an estimated mean. The formula for updating it is

$$a_t = a_{t-1} + t^{-1}(p_{t-1} - a_{t-1})$$

Substitute for the price using the ALM

$$\begin{aligned}a_t &= a_{t-1} + t^{-1}(\alpha + \beta a_{t-1} + \nu_{t-1} - a_{t-1}) \\ &= a_{t-1} + t^{-1}(\alpha + (\beta - 1)a_{t-1} + \nu_{t-1})\end{aligned}$$

The ODE thus takes the form

$$Q(t, a_{t-1}, \nu_t) = (\alpha + (\beta - 1)a_{t-1} + \nu_{t-1})$$

$$h(a) = \lim_{t\to\infty} E\left[(\alpha + (\beta - 1)a + \nu_{t-1})\right] = \alpha + (\beta - 1)a$$

$$\frac{da}{d\tau} = \alpha + (\beta - 1)a$$

We know that if $(\beta - 1) < 0$, then $a_t \to \bar{a}$. Since we assume $\beta < 1$, the model is E-stable and converges to the REE!

v) We now assume constant gains and replace the decreasing gains sequence $\{t^{-1}\}$ by constant gains $0 < \gamma \leq 1$. Write down the new updating formula for a_t. Substitute the ALM for the price and determine the process followed by a_t. Is it stationary?

Solution:

$$\begin{aligned}a_t &= a_{t-1} + \gamma(p_{t-1} - a_{t-1}) \\ &= a_{t-1} + \gamma(\alpha + \beta a_{t-1} + \nu_{t-1} - a_{t-1}) \\ &= \gamma\alpha + (1 - \gamma(1 - \beta))a_{t-1} + \gamma\nu_{t-1}\end{aligned}$$

So that $a_t \sim AR(1)$. As $(1 - \beta) < 1$ and $\gamma < 1$ imply $(1 - \gamma(1 - \beta)) < 1$, this process is stationary.

vi) Describe the process followed by the price! Is it stationary? Compute its variance.

Solution:

$$
\begin{aligned}
p_t &= \alpha + \beta p_{t+1}^e + \nu_t \\
p_t &= \alpha + \beta a_t + \nu_t \\
p_t &= \alpha + \beta\left(\gamma\alpha + \left(1-\gamma\left(1-\beta\right)\right)a_{t-1} + \gamma\nu_{t-1}\right) + \nu_t \\
&= \alpha + \beta\left(\gamma\alpha + \left(1-\gamma\left(1-\beta\right)\right)\left(\frac{1}{\beta}p_{t-1} - \frac{1}{\beta}\alpha - \frac{1}{\beta}\nu_{t-1}\right) + \gamma\nu_{t-1}\right) + \nu_t \\
&= \alpha + \beta\gamma\alpha + \left(1-\gamma\left(1-\beta\right)\right)p_{t-1} - \left(1-\gamma\left(1-\beta\right)\right)\alpha - \left(1-\gamma\left(1-\beta\right)\right)\nu_{t-1} + \beta\gamma\nu_{t-1} + \nu_t \\
&= \left(1-\gamma\left(1-\beta\right)\right)p_{t-1} + \alpha + \beta\gamma\alpha - \alpha + \alpha\gamma - \alpha\gamma\beta - \nu_{t-1} + \gamma\nu_{t-1} - \beta\gamma\nu_{t-1} + \beta\gamma\nu_{t-1} + \nu_t \\
&= \gamma\alpha + \left(1-\gamma\left(1-\beta\right)\right)p_{t-1} + \nu_t - \left(1-\gamma\right)\nu_{t-1}
\end{aligned}
$$

So that $p_t \sim ARMA(1,1)$. Since $\left(1-\gamma\left(1-\beta\right)\right) < 1$ and $\left(1-\gamma\right) < 1$, it is stationary.

To find the variance of the ARMA(1,1) process $y_t = \phi y_{t-1} + w_t + \theta w_{t-1}$, premultiply by $y_{t-\tau}$ and apply the expectations operator

$$
\begin{aligned}
E\left[y_{t-\tau}y_t\right] &= \phi E\left[y_{t-\tau}y_{t-1}\right] + E\left[y_{t-\tau}w_t\right] + \theta E\left[y_{t-\tau}w_{t-1}\right] \\
\gamma\left(\tau\right) &= \phi\gamma\left(\tau-1\right) + E\left[y_{t-\tau}w_t\right] + \theta E\left[y_{t-\tau}w_{t-1}\right]
\end{aligned}
$$

where $\gamma\left(\tau\right)$ is the auto-covariance and the last two terms are zero for $\tau > 1$. For $\tau = 0$ and $\tau = 1$ we have

$$
\begin{aligned}
\gamma\left(0\right) &= \phi\gamma\left(1\right) + \sigma_w^2 + \theta E\left[\left(\phi y_{t-1} + w_t + \theta w_{t-1}\right)w_{t-1}\right] \\
&= \phi\gamma\left(1\right) + \sigma_w^2 + \theta\phi\sigma_w^2 + \theta^2\sigma_w^2 \\
\gamma\left(1\right) &= \phi\gamma\left(0\right) + \theta\sigma_w^2
\end{aligned}
$$

Substituting $\gamma\left(1\right)$ in $\gamma\left(0\right)$ yields

$$
\begin{aligned}
\gamma\left(0\right) &= \phi\left(\phi\gamma\left(0\right) + \theta\sigma_w^2\right) + \sigma_w^2 + \theta\phi\sigma_w^2 + \theta^2\sigma_w^2 \\
\gamma\left(0\right) &= \frac{1 + 2\theta\phi + \theta^2}{1-\phi^2}\sigma_w^2
\end{aligned}
$$

So that we have for the price process

$$
\begin{aligned}
Var\left(p_t\right) &= \frac{1 - 2\left(1-\gamma\left(1-\beta\right)\right)\left(1-\gamma\right) + \left(1-\gamma\right)^2}{1 - \left(1-\gamma\left(1-\beta\right)\right)^2}\sigma_\nu^2 \\
&= \frac{1 - 2\left(1-\gamma+\gamma\beta\right)\left(1-\gamma\right) + 1 - 2\gamma + \gamma^2}{1 - \left(1-\gamma+\gamma\beta\right)^2}\sigma_\nu^2 \\
&= \frac{1 - 2 + 4\gamma - 2\gamma^2 - 2\gamma\beta + 2\gamma^2\beta + 1 - 2\gamma + \gamma^2}{1 - 1 + \gamma - \gamma\beta + \gamma - \gamma^2 + \gamma^2\beta - \gamma\beta + \gamma^2\beta - \gamma^2\beta^2}\sigma_\nu^2 \\
&= \frac{2\gamma - \gamma^2 - 2\gamma\beta + 2\gamma^2\beta}{2\gamma - 2\gamma\beta - \gamma^2 + 2\gamma^2\beta - \gamma^2\beta^2}\sigma_\nu^2 \\
&= \frac{\gamma\left(1 + 1 - 2\beta - \gamma + 2\gamma\beta\right)}{\gamma\left(1 + 1 - 2\beta - \gamma + 2\gamma\beta - \gamma\beta^2\right)}\sigma_\nu^2 \\
&= \frac{1 + \left(1-2\beta\right)\left(1-\gamma\right)}{1 + \left(1-2\beta\right)\left(1-\gamma\right) - \gamma\beta^2}\sigma_\nu^2
\end{aligned}
$$

Note that as soon as $\gamma > 0$, the variance of the price is bigger than the variance of the disturbance!

vii) In this last exercise you are asked to do some simulations. Use the following calibration: $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.2$, $a_0 = 3$. Do 1000 simulations (i.e. simulate 1000 observations).

First, simulate the process of the coefficient and the price when we have decreasing gains. What are the theoretical equilibrium values? Do the simulated processes converge to the theoretical values?
Second, simulate the process of the coefficient and the price when we have constant gains. Do the graphs of the coefficient estimates and of the price look like they converge to something? Do the correlograms of the processes confirm your answers to questions v) and vi)?
Compare the histograms of decreasing gains simulations with those of constant gains simulations. Do 10'000 simulations and compare again. What can you conclude asymptotically for the constant gain processes?

Solution:

Theoretical values for decreasing gains:

$$\frac{d}{d\tau}(a) = \alpha + (\beta - 1)a = 0 \Rightarrow a = \frac{-\alpha}{\beta - 1}$$

$$E[p_t] = a = \frac{-\alpha}{\beta - 1}$$

Programme code:

```
delete *
close few

create few u 1 10000

rndseed 123456789

scalar obs=10000

scalar alpha=0.8
scalar beta=0.7
scalar gamma=0.2

vector(obs) ad
ad(1)=3
vector(obs) pd

vector(obs) ac
ac(1)=3
vector(obs) pc
pc(1)=0

vector(obs) v
```

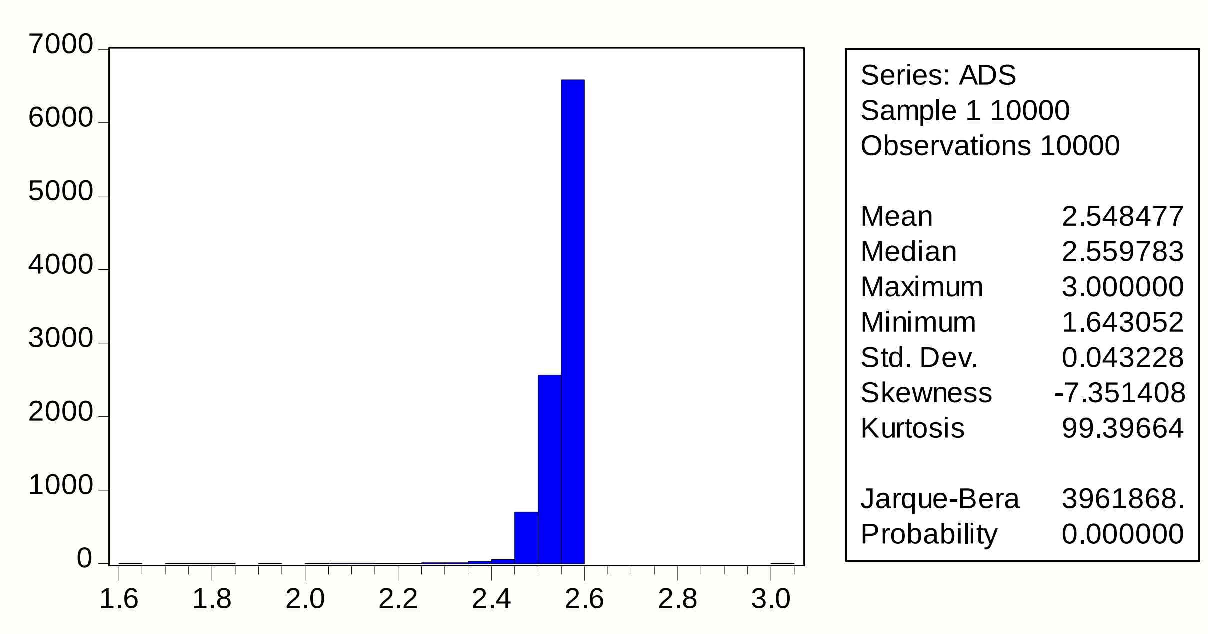
```
nrnd(v)

for !i=2 to obs
	ad(!i)=ad(!i-1)+((!i)^(-1))*(alpha+(beta-1)*ad(!i-1)+v(!i-1))
	pd(!i)=alpha+beta*ad(!i)+v(!i)
	ac(!i)=gamma*alpha+(1-gamma*(1-beta))*ac(!i-1)+gamma*v(!i-1)
	pc(!i)=alpha+beta*ac(!i)+v(!i)
next

mtos(ad,ads)
mtos(pd,pds)
mtos(ac,acs)
mtos(pc,pcs)
```

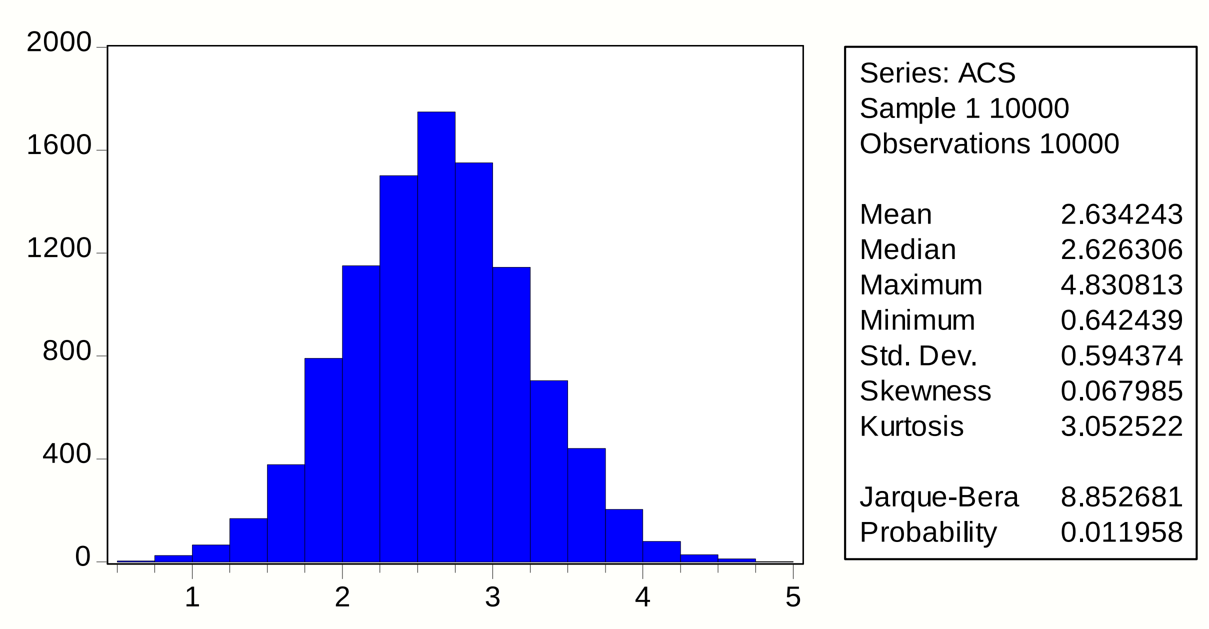
For 10000 obs:

Coefficient a, decreasing gains:



The mean is close to the theoretical value!

Coefficient a, constant gains:



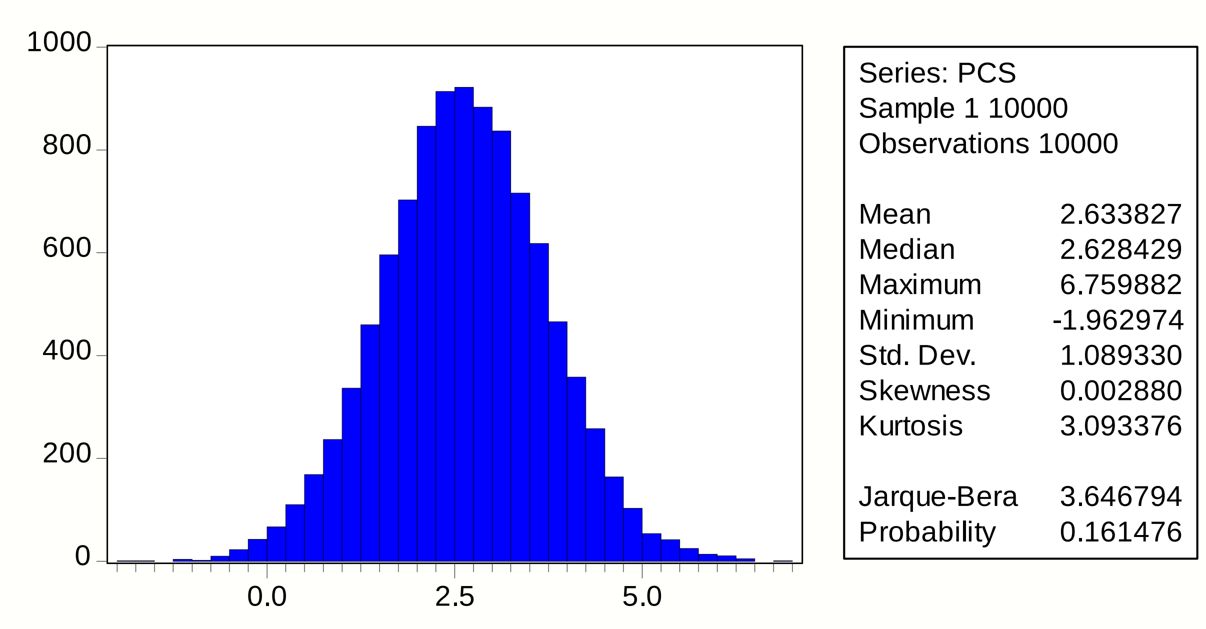
Asymptotically the AR(1) process has the same mean as the theoretical REE!

Sample: 1 10000
Included observations: 10000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.941	0.941	8858.4	0.000
		2	0.886	0.000	16704.	0.000
		3	0.834	0.002	23658.	0.000
		4	0.784	-0.006	29809.	0.000
		5	0.738	0.002	35255.	0.000
		6	0.695	0.008	40089.	0.000
		7	0.652	-0.020	44349.	0.000
		8	0.613	0.002	48107.	0.000
		9	0.576	0.005	51429.	0.000
		10	0.543	0.009	54376.	0.000
		11	0.510	-0.010	56978.	0.000
		12	0.481	0.015	59292.	0.000
		13	0.452	-0.008	61341.	0.000
		14	0.424	-0.011	63146.	0.000
		15	0.399	0.005	64740.	0.000
		16	0.374	-0.007	66142.	0.000

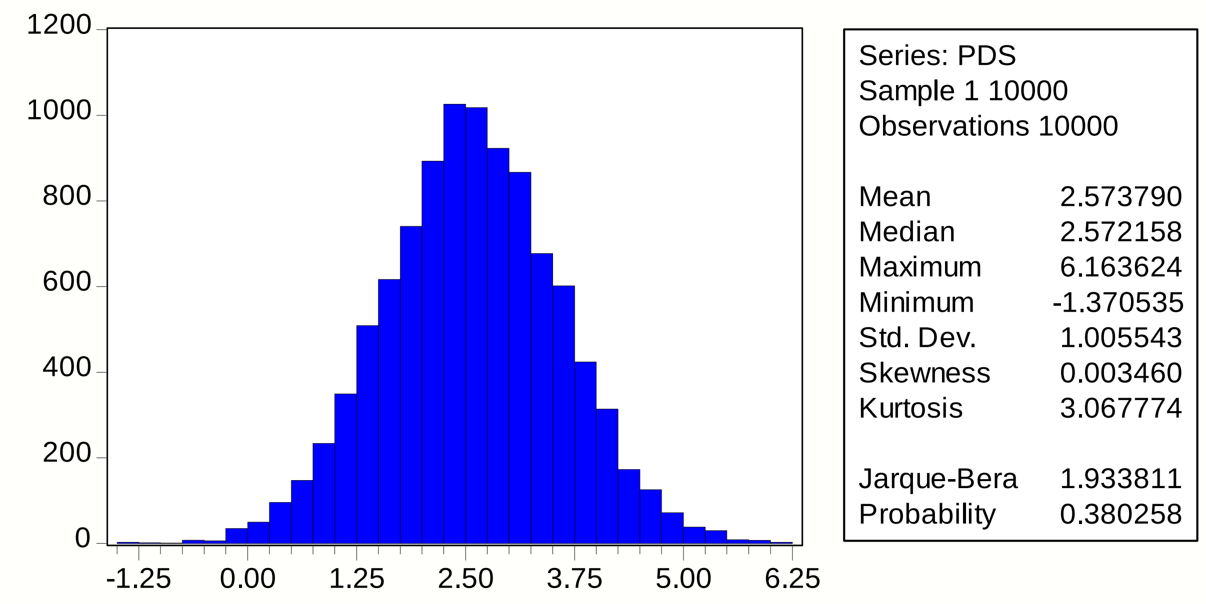
We clearly see the AR(1) pattern!

Price, decreasing gains



The mean is close to the theoretical value.

Price, constant gains



Looks very similar to decreasing gains price. Mean is close to theoretical value.

Sample: 1 10000
Included observations: 10000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.260	0.260	674.49	0.000
		2	0.243	0.188	1265.3	0.000
		3	0.234	0.149	1815.5	0.000
		4	0.215	0.108	2277.5	0.000
		5	0.198	0.081	2671.0	0.000
		6	0.206	0.087	3094.6	0.000
		7	0.177	0.048	3409.8	0.000
		8	0.164	0.035	3679.6	0.000
		9	0.153	0.027	3912.7	0.000
		10	0.157	0.038	4158.4	0.000
		11	0.130	0.009	4326.7	0.000
		12	0.139	0.028	4519.9	0.000
		13	0.132	0.023	4695.2	0.000
		14	0.112	0.003	4821.3	0.000
		15	0.114	0.013	4950.9	0.000
		16	0.110	0.013	5072.7	0.000

Typical ARMA(1,1) correlogram!