I) Title: Rational Expectations and Adaptive Learning

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III) Documentation:

- Basdevant, Olivier. (2003). Learning process and rational expectations: an analysis using a small macroeconomic model for New Zealand. *Discussion Paper Series*, Reserve Bank of New Zealand, DP2003/05.
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Rational Expectations and Adaptive Learning

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1. Introduction

The rational expectations hypothesis has often been criticized for being too strong an assumption. It has been suggested to relax this assumption by modelling the behaviour of economic agents as the behaviour of statisticians when they produce forecasts about the future state of the economy. This approach is called *adaptive learning* as the agents update their forecasting rule as new observations become available. Despite this bounded rationality of the agents, a rational expectations equilibrium might be learned by them. We will present the main ideas and results of the adaptive learning literature by applying them to the Lucas aggregate supply model.

2. The Lucas aggregate supply model

Consider the model based on Lucas (1973) with aggregate supply function given by

$$y_t = \bar{y} + \pi\left(p_t - p_t^e\right) + \zeta_t \tag{0.1}$$

where y_t denotes the log of output, $\bar{y}$ is the log of long run output, p_t the log of current prices, p_t^e the expectations of prices for time t based on information up to time $t-1$, ζ_t is a white noise shock and $\pi > 0$ is a parameter. The aggregate demand in logs is

$$m_t + v_t = p_t + y_t \tag{0.2}$$

where m_t is the money supply, following the policy rule

$$m_t = \bar{m} + \rho' w_{t-1} + u_t \tag{0.3}$$

and v_t is a velocity shock determined by

$$v_t = \mu + \gamma' w_{t-1} + \xi_t \tag{0.4}$$

where w_{t-1} is an $n \times 1$ vector of exogenous variables with $E\left[\mathbf{w}_t\right] = \mathbf{0}$, $E\left[\mathbf{w}_t \mathbf{w}_t'\right] = \mathbf{\Omega}$ and both u_t and ξ_t are white noise shocks.

Solving for the reduced form yields

$$\begin{aligned} p_t &= \left(1+\pi\right)^{-1}\left(m_t - \bar{y} + v_t + \pi p_t^e\right) \\ &= \left(1+\pi\right)^{-1}\left(\bar{m} + \mu - \bar{y}\right) + \pi\left(1+\pi\right)^{-1} p_t^e + \left(1+\pi\right)^{-1}\left(\rho + \gamma\right)' w_{t-1} \\ &\quad + \left(1+\pi\right)^{-1}\left(u_t + \xi_t - \zeta_t\right) \\ &= \varphi + \alpha p_t^e + \delta' w_{t-1} + \eta_t \end{aligned} \tag{0.5}$$

Note that we have by definition $0 < \alpha = \pi\left(1+\pi\right)^{-1} < 1$.

The rational expectations assumption uses $p_t^e = E\left[p_t \middle| \Omega_{t-1}\right]$ where Ω_{t-1} denotes the information set and by applying the conditional expectation operator to (0.5) we get

$$E\left[p_t \middle| \Omega_{t-1}\right] = \left(1-\alpha\right)^{-1}\left(\varphi + \delta' w_{t-1}\right) \tag{0.6}$$

Another implication of the rational expectations assumption is that $p_t - E\left[p_t \middle| \Omega_{t-1}\right] = \varepsilon_t$ where ε_t is a white noise error. We can obtain the rational expectations equilibrium (REE) from (0.6)

$$p_t = (1-\alpha)^{-1}\varphi + (1-\alpha)^{-1}\delta' w_{t-1} + \varepsilon_t$$
$$= \bar{a} + \bar{b}' w_{t-1} + \varepsilon_t \tag{0.7}$$

We conclude that this model has a unique REE given by (0.7).

3.) Econometric Learning

We now relax the hypothesis of rational expectations and assume that agents form their expectations by *econometric learning*. Knowing that the Lucas model has a unique REE, it will be of most interest to know whether it is learnable or not in the sense that it converges under econometric learning to the REE. Suppose that firms believe that prices follow the process

$$p_t = a + b' w_{t-1} + \varepsilon_t \tag{0.8}$$

corresponding to the RRE (0.7), but that the true parameters $\bar{a}$ and $\bar{b}$ are unknown to them. Possible reasons might either be that firms do not know the structure of the economy but correctly assume that p_t depends linearly on w_{t-1}, or that they know the structure but not the structural parameters π, ρ and γ. Equation (0.8) is called the *perceived law of motion* (PLM) and firms use least squares regression to estimate the parameters of it. Denote by a_{t-1} and b_{t-1} the LS estimates based on information available at time $t-1$ and computed with the standard LS formula

$$\begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} = \left(\sum_{i=1}^{t-1} z_{i-1} z_{i-1}' \right)^{-1} \left(\sum_{i=1}^{t-1} z_{i-1} p_i \right) \tag{0.9}$$

where $z_i' = (1 \quad w_i')$.

The expected price is then given by the forecast

$$p_t^e = a_{t-1} + b_{t-1}' w_{t-1} \tag{0.10}$$

Equations (0.5), (0.9) and (0.10) fully specify a dynamic system. At $t-1$ firms produce estimates a_{t-1} and b_{t-1}, which they use to forecast p_t^e. At the beginning of time t, given w_{t-1} and η_t, p_t is determined by (0.5). Firms then produce estimates a_t and b_t...

The question of interest now is if $\lim_{t\to\infty} a_t = \bar{a}$ and $\lim_{t\to\infty} b_t = \bar{b}$.

4.) Expectational Stability

First we need to make sure that the REE is stable under learning. In our example we assume that the agents use the PLM (0.8) to form their price expectations. Note that we usually take the PLM to be of the same form as the REE of interest. If we substitute the price forecast back in (0.5), we can solve for the *actual law of motion* (ALM)

$$p_t = (\varphi + \alpha a) + (\alpha b + \delta)' w_{t-1} + \eta_t \tag{0.11}$$

The ALM describes the stochastic process followed by the economy if forecasts are made by using the PLM. This implicitly defines the *mapping T from the PLM to the ALM* for the coefficients a and b

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varphi + \alpha a \\ \delta + \alpha b \end{pmatrix} \quad (0.12)$$

Note that the unique REE is the unique fixed point of the T-map. Consider the differential equation

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \quad (0.13)$$

The REE is said to be *expectationally stable* – or *E-stable* – if the REE is locally asymptotically stable under (0.13). E-stability thus determines the stability of the REE under the least squares learning rule which is used to gradually adjust the PLM parameters a and b in the direction of the implied ALM parameters. The REE $\left(\bar{a}, \bar{b}'\right)'$ is stable if small displacements from $\left(\bar{a}, \bar{b}'\right)'$ are returned to $\left(\bar{a}, \bar{b}'\right)'$ under the learning rule.

Combining (0.12) and (0.13) we can determine the E-stability of the Lucas model

$$\begin{aligned} \frac{da}{d\tau} &= \varphi + (\alpha - 1)a \\ \frac{db_i}{d\tau} &= \delta_i + (\alpha - 1)b_i \quad \text{for} \quad i = 1, \ldots n \end{aligned} \quad (0.14)$$

The equations in (0.14) imply that the model is E-stable if and only if $\alpha < 1$. By recalling $0 < \alpha = \pi(1+\pi)^{-1} < 1$, we can conclude that the Lucas model is E-stable.

5.) Convergence

5.1) Recursive least squares

The next question of interest is whether the model converges to the REE under least squares learning. We assume that agents use recursive least squares (RLS) to update their estimates a_{t-1} and b_{t-1} as new observations of p_t and w_{t-1} become available. We use another version of the RLS formulae we have seen in this class so far.

Let $\mathbf{P}_t = \left(p_1 \quad \ldots \quad p_t\right)'$ be the $t \times 1$ vector of endogenous variables, let $\mathbf{Z}_t = \left(\mathbf{z}_0 \quad \ldots \quad \mathbf{z}_{t-1}\right)'$ where $z'_{t-1} = \left(1 \quad \mathbf{w}'_{t-1}\right)$ be the $t \times k$ matrix of explanatory variables and $\boldsymbol{\varphi}_t = \left(a_t \quad \mathbf{b}'_t\right)'$ the $k \times 1$ vector containing estimated coefficients. From the standard LS formula for a regression of p_t on $\mathbf{z}_{t-1}$ (cf (0.8)) we can derive the updating equation for $\boldsymbol{\varphi}_t$ as follows

$$\begin{aligned} \boldsymbol{\varphi}_t &= \left(\mathbf{Z}'_t\mathbf{Z}_t\right)^{-1}\mathbf{Z}'_t\mathbf{P}_t \\ &= \left(\mathbf{Z}'_t\mathbf{Z}_t\right)^{-1}\left(\mathbf{Z}'_{t-1}\mathbf{P}_{t-1} + \mathbf{z}_{t-1}p_t\right) = \left(\mathbf{Z}'_t\mathbf{Z}_t\right)^{-1}\left(\left(\mathbf{Z}'_{t-1}\mathbf{Z}_{t-1}\right)\boldsymbol{\varphi}_{t-1} + \mathbf{z}_{t-1}p_t\right) \\ &= \left(\mathbf{Z}'_t\mathbf{Z}_t\right)^{-1}\left(\left(\mathbf{Z}'_t\mathbf{Z}_t - \mathbf{z}_{t-1}\mathbf{z}'_{t-1}\right)\boldsymbol{\varphi}_{t-1} + \mathbf{z}_{t-1}p_t\right) \\ &= \boldsymbol{\varphi}_{t-1} + \left(\mathbf{Z}'_t\mathbf{Z}_t\right)^{-1}\mathbf{z}_{t-1}\left(p_t - \mathbf{z}'_{t-1}\boldsymbol{\varphi}_{t-1}\right) \\ &= \boldsymbol{\varphi}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{z}_{t-1}\left(p_t - \mathbf{z}'_{t-1}\boldsymbol{\varphi}_{t-1}\right) \end{aligned} \quad (0.15)$$

where $\mathbf{R}_t = t^{-1}\left(\mathbf{Z}'_t\mathbf{Z}_t\right)$ denotes the moment matrix for $\mathbf{z}$. The updating formula for $\mathbf{R}_t$ is given by

$$
\begin{aligned}
\mathbf{R}_t &= t^{-1}\left(\mathbf{Z}_t'\mathbf{Z}_t\right) = t^{-1}\left(\mathbf{Z}_{t-1}'\mathbf{Z}_{t-1} + z_{t-1}z_{t-1}'\right) \\
&= \frac{t\mathbf{Z}_{t-1}'\mathbf{Z}_{t-1} + (t-1)\, z_{t-1}z_{t-1}' - \mathbf{Z}_{t-1}'\mathbf{Z}_{t-1}}{t(t-1)} \\
&= \mathbf{R}_{t-1} + t^{-1}\left(z_{t-1}z_{t-1}' - \mathbf{R}_{t-1}\right)
\end{aligned} \tag{0.16}
$$

Recalling (0.5) and (0.10) we have

$$
\begin{aligned}
p_t &= \left(\varphi + \alpha a_{t-1}\right) + \left(\delta + \alpha b_{t-1}\right)' w_{t-1} + \eta_t \\
&= T\left(\boldsymbol{\varphi}_{t-1}\right)' \mathbf{z}_t + \eta_t
\end{aligned} \tag{0.17}
$$

Combining (0.15) and (0.16) with (0.17) we get the *stochastic resursive system*

$$
\boldsymbol{\varphi}_t = \boldsymbol{\varphi}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{z}_{t-1}\left(\mathbf{z}_{t-1}'\left(T\left(\boldsymbol{\varphi}_{t-1}\right) - \boldsymbol{\varphi}_{t-1}\right) + \eta_t\right) \tag{0.18}
$$

$$
\mathbf{R}_t = \mathbf{R}_{t-1} + t^{-1}\left(z_{t-1}z_{t-1}' - \mathbf{R}_{t-1}\right) \tag{0.19}
$$

It remains to determine whether this stochastic recursive system converges as $t \to \infty$. We would like to find that $\boldsymbol{\varphi}_t \to \overline{\boldsymbol{\varphi}}$ because from $T\left(\overline{\boldsymbol{\varphi}}\right) = \overline{\boldsymbol{\varphi}}$ – the unique REE is the unique fixed point of the map – we could then conclude that the price process converges to the REE!

At this point it becomes most obvious where the bounded rationality of the agents comes from: The true process followed by p_t, which is given in (0.17), has time varying parameters, but the agents are estimating a model (0.8) with constant parameters. This model misspecification makes the agents not to act fully rational. Note however that if the learned coefficients converge to the REE this difference disappears in the limit.

5.2) Stochastic recursive algorithms

We first show some general results and then apply those to the Lucas model.
Consider the stochastic recursive algorithm (SRA)

$$
\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \gamma_t Q\left(t, \boldsymbol{\theta}_{t-1}, \mathbf{x}_t\right) \tag{0.20}
$$

where $\boldsymbol{\theta}_t$ is the vector of estimates, γ_t is a deterministic sequence of gains, and $\mathbf{x}_t$ is the state vector which may either follow an exogenous process or a VAR process where the coefficients depend on $\boldsymbol{\theta}_{t-1}$. The function $Q(\cdot)$ determines how $\boldsymbol{\theta}_{t-1}$ is updated as the observation of the last period becomes available. Stochastic approximation results show that the behaviour of a SRA is well approximated by an ordinary differential equation (ODE) for large t

$$
\frac{d\boldsymbol{\theta}}{d\tau} = h\left(\boldsymbol{\theta}(\tau)\right) = \lim_{t\to\infty} E\left[Q\left(t, \boldsymbol{\theta}, \mathbf{x}_t\right)\right] \tag{0.21}
$$

if that limit exists. Possible limit points of the SRA correspond to locally stable equilibria of the ODE. Under suitable assumptions, if $\overline{\boldsymbol{\theta}}$ is a locally stable equilibrium point of the ODE, then $\overline{\boldsymbol{\theta}}$ is a possible point of convergence of the SRA. If $\overline{\boldsymbol{\theta}}$ is not a locally stable quliibrium point of the ODE, then $\overline{\boldsymbol{\theta}}$ is not a possible point of convergence of the SRA, i.e. $\boldsymbol{\theta}_t \to \overline{\boldsymbol{\theta}}$ with probability 0.

The necessary technical assumptions in order to obtain this convergence conditions are in particular that we need regularity assumptions on $Q(\cdot)$, conditions on the rate at which $\gamma_t \to 0$, and assumptions on the properties of the stochastic process followed by $\mathbf{x}_t$. We just note that all those conditions are met by the Lucas model.

Parenthesis: Short review of ODE properties:
- $\bar{\boldsymbol{\theta}}$ is an equilibrium point of the ODE $d\boldsymbol{\theta}/d\tau = h(\boldsymbol{\theta})$ if $h(\bar{\boldsymbol{\theta}}) = 0$.
- $\bar{\boldsymbol{\theta}}$ is locally stable if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\left|\boldsymbol{\theta}(\tau) - \bar{\boldsymbol{\theta}}\right| < \varepsilon$ for all $\left|\boldsymbol{\theta}(0) - \bar{\boldsymbol{\theta}}\right| < \delta$.
- $\bar{\boldsymbol{\theta}}$ is locally asymptotically stable if it is locally stable and $\boldsymbol{\theta}(\tau) \to \bar{\boldsymbol{\theta}}$ for all $\boldsymbol{\theta}(0)$ in the neighbourhood of $\bar{\boldsymbol{\theta}}$.
- $\bar{\boldsymbol{\theta}}$ is locally unstable if it is not locally stable.

Further recall that the Jacobian $Dh(\bar{\boldsymbol{\theta}})$ of $h(\bar{\boldsymbol{\theta}})$ provides information on the local stability of $\bar{\boldsymbol{\theta}}$:
- If all eigenvalues of $Dh(\bar{\boldsymbol{\theta}})$ have negative real parts, then $\bar{\boldsymbol{\theta}}$ is a locally stable equilibrium point of $d\boldsymbol{\theta}/d\tau = h(\boldsymbol{\theta})$.
- If some eigenvalue of $Dh(\bar{\boldsymbol{\theta}})$ has a positive real part, then $\bar{\boldsymbol{\theta}}$ is not a locally stable equilibrium point of $d\boldsymbol{\theta}/d\tau = h(\boldsymbol{\theta})$.

5.3) Application to the Lucas model

First, we need to put the SRA given by equations (0.18)-(0.19) in standard form

$$\boldsymbol{\varphi}_t = \boldsymbol{\varphi}_{t-1} + t^{-1}\mathbf{S}_{t-1}^{-1}\mathbf{z}_{t-1}\left(\mathbf{z}'_{t-1}\left(T(\boldsymbol{\varphi}_{t-1}) - \boldsymbol{\varphi}_{t-1}\right) + \eta_t\right) \tag{0.22}$$

$$\mathbf{S}_t = \mathbf{S}_{t-1} + t^{-1}\frac{t}{t+1}\left(z_{t-1}z'_{t-1} - \mathbf{S}_{t-1}\right) \tag{0.23}$$

where we had to change notation $\mathbf{S}_{t-1} = \mathbf{R}_t$ since the standard form allows only lagged values on the RHS of the equations (0.22) and (0.23)

Let $\boldsymbol{\theta}_t = vec\begin{pmatrix}\boldsymbol{\varphi}_t & \mathbf{S}_t\end{pmatrix}$, $\mathbf{x}_t = \begin{pmatrix}1 & \mathbf{w}'_t & \mathbf{w}'_{t-1} & \eta_t\end{pmatrix}'$ and $\gamma_t = t^{-1}$ then we can write the two components of the function $Q(t, \boldsymbol{\theta}_{t-1}, \mathbf{x}_t)$ as

$$\begin{aligned} Q_\phi(t, \boldsymbol{\theta}_{t-1}, \mathbf{x}_t) &= \mathbf{S}_{t-1}^{-1}\mathbf{z}_{t-1}\left(\mathbf{z}'_{t-1}\left(T(\boldsymbol{\varphi}_{t-1}) - \boldsymbol{\varphi}_{t-1}\right) + \eta_t\right) \\ Q_S(t, \boldsymbol{\theta}_{t-1}, \mathbf{x}_t) &= vec\left(\frac{t}{t+1}\left(z_{t-1}z'_{t-1} - \mathbf{S}_{t-1}\right)\right) \end{aligned} \tag{0.24}$$

Next, compute the associated ODE. To do so we have to fix a value for $\boldsymbol{\theta}$ and take the expectation over $\mathbf{x}_t$ to get

$$\begin{aligned} h_\phi(\boldsymbol{\varphi}, \mathbf{S}) &= \lim_{t\to\infty} E\left[\mathbf{S}^{-1}\mathbf{z}_{t-1}\left(\mathbf{z}'_{t-1}\left(T(\boldsymbol{\varphi}) - \boldsymbol{\varphi}\right) + \eta_t\right)\right] \\ h_S(\boldsymbol{\varphi}, \mathbf{S}) &= \lim_{t\to\infty}\frac{t}{t+1} E\left[\left(z_{t-1}z'_{t-1} - \mathbf{S}\ \right)\right] \end{aligned} \tag{0.25}$$

Let $E[\mathbf{z}_t\mathbf{z}'_t] = E[\mathbf{z}_{t-1}\mathbf{z}'_{t-1}] = \begin{bmatrix}1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}\end{bmatrix} = \mathbf{M}$, note that $E[\mathbf{z}_{t-1}\eta_t] = \mathbf{0}$ and $\lim_{t\to\infty}\frac{t}{t+1} = 1$, then

$$
\begin{aligned}
h_{\phi}\left(\boldsymbol{\varphi},\mathbf{S}\right) &= \mathbf{S}^{-1}\mathbf{M}\left(T\left(\boldsymbol{\varphi}\right)-\boldsymbol{\varphi}\right) \\
h_{S}\left(\boldsymbol{\varphi},\mathbf{S}\right) &= \mathbf{M}-\mathbf{S}
\end{aligned}
\tag{0.26}
$$

The associated ODE is therefore

$$
\begin{aligned}
\frac{d\boldsymbol{\varphi}}{d\tau} &= \mathbf{S}^{-1}\mathbf{M}\left(T\left(\boldsymbol{\varphi}\right)-\boldsymbol{\varphi}\right) \\
\frac{d\mathbf{S}}{d\tau} &= \mathbf{M}-\mathbf{S}
\end{aligned}
\tag{0.27}
$$

This system is recursive and the second equation is globally stable, thus $\mathbf{S}\to\mathbf{M}$ from any starting point. This implies $\mathbf{S}^{-1}\mathbf{M}\to\mathbf{I}$ if $\mathbf{S}$ is invertible along the path and we therefore only have to look at $T\left(\boldsymbol{\varphi}\right)-\boldsymbol{\varphi}$ in order to find out if the system (0.27) is stable. From the definition (0.12) we find

$$
T\left(\boldsymbol{\varphi}\right)-\boldsymbol{\varphi} = \begin{pmatrix}\varphi \\ \delta\end{pmatrix} + \left(\alpha-1\right)\mathbf{I}\boldsymbol{\varphi}
\tag{0.28}
$$

which is a linear differential equation with coefficient matrix $\left(\alpha-1\right)\mathbf{I}$. Note that all eigenvalues of $\left(\alpha-1\right)\mathbf{I}$ have negative real parts if $\alpha<1$. Since this is the case we find that $\overline{\boldsymbol{\varphi}}$ is a globally stable equilibrium point of (0.28) and the SRA results therefore imply that for the SRA (0.22)-(0.23) $\left(\boldsymbol{\varphi}_t,\mathbf{S}_t\right)\to\left(\overline{\boldsymbol{\varphi}},\mathbf{M}\right)$ with probability 1.

The Lucas model is found to be E-stable under learning and it will always converge to the REE under learning.