Empirical Macroeconomics

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Autumn term 2011

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Topic 4: Models of Optimising Agents

- Introduction
- Solution of Optimal Control Problem by Lagrange Multipliers
- Steady-State Solution
- Dynamic Programming

Introduction

- Up to this point the methods used helped only to understand the impact of selected policies by governments
- Now we want to find an answer to the question: „Which is the ‚best‘ measures?“
- A government usually has the power to change or manipulate certain variables of the system. These are referred to as **control variables**

Feedback control equation

- The aim is to find a **feedback control equation** of the form

$$X_t = GZ_{t-1} + g \qquad (1)$$

  where current policy depends on future observations
- The term feedback is attached to indicate that current policy determines future policy

Underlying model

- The underlying model is linear

$$Z_t = BZ_{t-1} + CX_t + b + u_t \qquad (2)$$

- A control rule, such as (1) can be combined with (2) to get

$$Z_t = (B + CG)Z_{t-1} + (CG + b) + u_t \qquad (3)$$

- The important question now is, how should any outcome be judged, so to make them comparable.
- A – mathematically – convenient form is a quadratic function

Objective Function

- This is done by the **objective function**, also called the **welfare function.**
- A widely used – mathematically convenient – objective function is the quadratic function
- Assume that (2) is already a reduced form VAR, so that the vector Z_t includes all variables of the model
- The welfare function can be written as

$$E_0 W = E_0 \sum_{t=1}^{T} (Z_t - a_t)' K (Z_t - a_t) \qquad (4)$$

- a_t is the target vector and K a positive semi-definite matrix

Objective Function (cont'd)

- **An optimal control problem is to minimise the expected welfare loss (4), given the econometric model (2)**
- Four remarks about the objective function
  - Favourite candidates for inclusion in the welfare function are unemployment rate, inflation and output gap
  - Controlling the level of a variable is different from controlling ist first difference. The former does not penalise period-to-period changes whereas the second does
  - A positive deviation is treated the same as a negative deviation
  - The objective function is additive, thus it ignores the variance over time

Solution to the Optimal Control Problem

- In order to solve the optimal control problem, it is decomposed into two parts: the deterministic and the stochastic control problem.
- The deterministic problem uses the model

$$\overline{Z}_t = B\overline{Z}_{t-1} + C\overline{X}_t + b_t \ , \ (\overline{Z}_0 = Z_0) \quad (5)$$

which is obtained by setting the random disturbance equal to its mean

- The stochastic model is

$$Z_t^* = BZ_{t-1}^* + CX_t^* + u_t \ , \ (Z_0^* = 0) \quad (6)$$

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Solution to the Optimal Control Problem (cont'd)

- Accordingly, the welfare function can be decomposed into

$$E(W) = \sum_{t=1}^{T} (\overline{Z}_t - a_t)' K (\overline{Z}_t - a_t) + E \sum_{t=1}^{T} Z_t{}^{*\prime} K Z_t{}^{*} = W_1 + E(W_2) \qquad (7)$$

- First the deterministic problem will be solved to minimise W_1 with respect to $\overline{X}_t$
- Then, the stochastic problem will be solved to minimise $E(W_2)$ with respect to $X_t{}^{*}$
- When the two separate solutions are found, the policy variables are set equal to the sum of the two.

Solution of the Deterministic Control Problem by use of Lagrange multipliers

- Introduce the vector λ_t of Lagrange multipliers and differentiate the Lagrangian expression

$$\mathcal{L}_1 = \frac{1}{2}\sum_{t=1}^{T}(\overline{Z}_t - a_t)' K_T (\overline{Z}_t - a_t) - \sum_{t=1}^{T}\lambda_t' \left(\overline{Z}_t - B\overline{Z}_{t-1} - C\overline{X}_t - b_t\right) \quad (8)$$

- The following derivatives are obtained

$$\frac{\delta \mathcal{L}_1}{\delta \overline{Z}_t} = K_T(\overline{Z}_t - a_t) - \lambda_t - B'\lambda_{t+1} = 0 \qquad (t = 1, \ldots, T; \lambda_{t+1} = 0) \qquad (9)$$

$$\frac{\delta \mathcal{L}_1}{\delta \overline{X}_t} = C'\lambda_t = 0 \qquad (t = 1, \ldots, T) \qquad (10)$$

$$\frac{\delta \mathcal{L}_1}{\delta \lambda_t} = -\overline{Z}_t + B\overline{Z}_{t-1} + C\overline{X}_t + b_t = 0 \qquad (t = 1, \ldots, T) \qquad (11)$$

- The way of solving the problem is to start at $t = T$ and repeat the following three steps:

1. (9) is used to express λ_t as a function of $\bar{Z}_t$.
2. The result, together with (10) and (11), is used to solve for $\bar{X}_t$
3. The results of the first two steps together with (11) are used to express $\bar{Z}_t$ and λ_t as linear functions of $\bar{Z}_{t-1}$. Using the last linear function, express λ_{t-1} as a linear function of $\bar{Z}_{t-1}$, and begin again with step one for the next round.

- In the first step

$$\lambda_T = K_T \bar{Z}_T - K_T a_T - B' \lambda_{T+1} = H_T \bar{Z}_T - h_T \tag{12}$$

where

$$H_T = K_T \tag{13}$$

$$h_T = K_T a_T \tag{14}$$

- The second step yields

$$\bar{X}_T = G_T \bar{Z}_{T-1} + g_T \tag{15}$$

where

$$G_T = -(C' H_T C)^{-1} C' H_T B \tag{16}$$

$$g_T = -(C' H_T C)^{-1} C' (H_T b_T - h_T) \tag{17}$$

- In the third step we use (11) and (15) to solve for $\bar{Z}_T$ as a function of $\bar{Z}_{T-1}$

$$\bar{Z}_T = (B + CG_T)\bar{Z}_{T-1} + Cg_T + b_T \tag{18}$$

- The result can be applied to (12) to express λ_T also as a function of $\bar{Z}_{T-1}$

$$\lambda_T = H_T(B + CG_T)\bar{Z}_{T-1} + H_T(Cg_T + b_T) - h_T \tag{19}$$

- Having solved for λ_T in terms of $\bar{Z}_{T-1}$, substitute (19) into (9) to obtain an equation analogous to (12) in the first step

$$\lambda_{T-1} = K(\bar{Z}_{T-1} - a_{T-1}) + B'\lambda_T = H_{T-1}\bar{Z}_{T-1} - h_{T-1} \tag{20}$$

where

$$H_{T-1} = K_{T-1} + B'H_T(B + CG_T) \tag{21}$$

$$h_{T-1} = K_{T-1}a_{T-1} - B'H_T(b_T + C_T g_T) + B'h_T \tag{22}$$

- Note that the solution of the optimal $\bar{X}_t$ is a linear function of $\bar{Z}_{t-1}$. Note also that the coefficients G and g are obtained by solving backwards in time two sets of difference equations.

- For G, this is

$$H_{t-1} = K_{t-1} + B_t' H_t (B_t + C_t G_t)$$

$$= K_{t-1} + B_t' H_t B_t - B_t' H_t C_t (C_t' H_t C_t)^{-1} C_t' H_t B_t \qquad (23)$$

also known as the matrix Riccati equation.

- To obtain g, one also needs

$$
\begin{aligned}
h_{t-1} &= K_{t-1}a_{t-1} - B_t'H_t(b_t + C_t g_t) + B_t'h_t \\
&= K_{t-1}a_{t-1} - B_t'H_t b_t + B_t'H_t C_t (C_t'H_t C_t)^{-1} C_t'(H_t b_t - h_t) + B_t'h_t \\
&= K_{t-1}a_{t-1} + (B_t + C_t G_t)'(h_t - H_t b_t) \qquad (24)
\end{aligned}
$$

- The difference equations are easy to solve, as the include only matrix operations.

Solution of Stochastic Control by Lagrange Multipliers

- The second part is to solve the stochastic control problem in order to minimise $E(W_2)$, given the system (7)
- We search for the optimal matrix G_t in the linear equation

$$X_t^* = G_t Z_{t-1}^* \tag{25}$$

- The problem is rewritten in terms of the (nonstationary) covariance matrices $E(Z_t^* Z_t^{*\prime})$ of the vectors Z_t^* and utilise a set of restrictions on these covariance matrices to form a Lagrange expression.
- The objective function is written as

$$E(W_2) = E\sum_{t=1}^{T} \mathrm{tr}(Z_t^{*\prime} K_t Z_t^*) = E\sum_{t=1}^{\mathrm{T}} \mathrm{tr}(K_t Z_t^* Z_t^{*\prime}) = \sum_{t=1}^{\mathrm{T}} \mathrm{tr} K_t (E(Z_t^* Z_t^{*\prime})) \quad (26)$$

where **tr**, or trace, is the sum of diagonal elements of a matrix and use is made of the property $\mathrm{tr}(BG) = \mathrm{tr}(GB)$

- Now, substitute the control rule (25) into (6) to find a set of restrictions on the covariance matrix

$$Z_t^* = (B_t + C_t G_t) Z_{t-1}^* + u_t$$

$$= R_t Z_{t-1}^* + u_t, \qquad (Z_0^* = 0) \tag{27}$$

- Postmultiply (27) by $Z_t^{*\prime}$ and take expectation.

$$E(Z_t^* Z_t^{*\prime}) = R_t E(Z_{t-1}^* Z_t^{*\prime}) + E(u_t Z_t^{*\prime})$$

$$= R_t E(Z_{t-1}^* Z_t^{*\prime}) + V \tag{28}$$

because by (27) $E(u_t Z_t^{*\prime}) = E(u_t u_t^{\prime}) = V$

- Transpose equation (27) and premultiply the result by Z_{t-1}^{*} and take expectation yields

$$E(Z_{t-1}^{*}Z_{t}^{*\prime}) = E(Z_{t-1}^{*}Z_{t-1}^{*}{}')R_{t}' + \underbrace{E(Z_{t-1}^{*}u_{t}')}_{=0} \qquad (29)$$

- Substitution of (29) into (28) gives the desired difference equation in $E(Z_{t}^{*}Z_{t}^{*\prime}) = \Gamma(t,0) = \Gamma_{\cdot t}$

$$E(Z_{t}^{*}Z_{t}^{*\prime}) = R_{t}E(Z_{t-1}^{*}Z_{t-1}^{*}{}')R_{t}' + V \qquad (30)$$

or

$$\Gamma_{\cdot t} = (B_{t} + C_{t}G_{t})\Gamma_{\cdot t-1}(B_{t} + C_{t}G_{t})' + V$$

- To incorporate (30) as a set of constraints for $t = 1,\ldots,T$ in a Lagrange expression for minimising (26) a $p \times p$ matrix H_t of Lagrange multipliers is introduced for each constraint
- Write the Lagrange expression as

$$\mathcal{L}_2 = \sum_{t=1}^{T} \operatorname{tr}(K_t \Gamma_{\cdot t}) - \sum_{t=1}^{T} \operatorname{tr}\left\{ H_t \left[\Gamma_{\cdot t} - V - (B_t + C_t G_t)\Gamma_{\cdot t-1}(B_t + C_t G_t)' \right] \right\} \quad (31)$$

- The variables are the elements of $G_t, \Gamma_{\cdot t},$ and H_t

- The derivatives of $\mathcal{L}_2$ are

$$\frac{\partial \mathcal{L}_2}{\partial G_t} = 2C_t'H_tB_t\Gamma_{\cdot t-1} + 2C_t'H_tC_tG_t\Gamma_{\cdot t-1} = 0 \qquad t = 1,\ldots,T \tag{32}$$

$$\frac{\partial \mathcal{L}_2}{\partial \Gamma_{\cdot t}} = K_t - H_t + (B_{t+1} + C_{t+1}G_{t+1})'H_{t+1}(B_{t+1} + C_{t+1}G_{t+1}) = 0 \tag{33a}$$

$$t = 1,\ldots,T-1$$

$$\frac{\partial \mathcal{L}_2}{\partial \Gamma_{\cdot T}} = K_T - H_T = 0 \tag{33b}$$

$$\frac{\partial \mathcal{L}_2}{\partial H_t} = -\left[\Gamma_{\cdot t} - V - (B_t + C_tG_t)\Gamma_{\cdot t-1}(B_t + C_tG_t)'\right] = 0 \qquad t = 1,\ldots,T \tag{34}$$

- Equations (32), (33) and (34) provide a set of necessary conditions for the unknowns $G_t, \Gamma_{\cdot t}$, and H_t
- Equation (32) can be satisfied by choosing

$$G_t = -(C_t' H_t C_t)^{-1} C_t' H_t B_t \quad (35)$$

- Equation (33) is equivalent to

$$H_t = K_t + (B_{t+1} + C_{t+1} G_{t+1})' H_{t+1} (B_{t+1} + C_{t+1} G_{t+1}) \quad (36)$$

- Therefore the unknowns G_t and H_t can be found by using the initial condition $H_T = K_T$ from (33b) and solving (35) and (36) alternately backward in time for $t = T, T-1, \ldots, 1$

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Equality of deterministic and stochastic solution

- It is interesting to observe that the coefficients G_t in the optimal feedback equations $X_t^* = G_t Z_{t-1}^*$ for controlling the variances of random deviations Z_t^* are identical with the coefficients of G_t in the optimal feedback control equations $\bar{Z}_t = B\bar{Z}_{t-1} + C\bar{Z}_t + b_t$ for steering the means $\bar{Z}_t$ to the targets a_t obtained previously for the deterministic control problem.
- Equation (36) is easily shown to be the same as (23) at $t = T$
- Having solved for the optimum G_t and H_t , we can find the covariance matrices $\Gamma_{\cdot t}$ by (34), namely

$$\Gamma_{\cdot t} = V + (B_t + C_t G_t)\Gamma_{\cdot t-1}(B_t + C_t G_t)' \quad t = 1, \ldots, T \qquad (37)$$

- Equation (37) is solved forward in time from t = 1 to t = T, using the initial condition $\Gamma_{\cdot 0} = E(Z_0^* Z_0^{*\prime}) = 0$ which is due to the definition $\bar{Z}_0 = Z_0$ or $Z_0^* = 0$

The Combined Solution and the Minimum Expected Loss

- Combining the solution of the two previous sections for the deterministic and the stochastic parts of the control problem gives the optimal control equation for X_t

$$X_t = \overline{X}_t + X_t^* = G_t \overline{Z}_{t-1} + g_t + G_t Z_{t-1}^*$$
$$= G_t Z_{t-1} + g_t \qquad (38)$$

- The matrices G_t are computed using (16) and (21) or equivalently by using (35) and (36). The vectors g_t are computed by (17) and (22). In the computations the targets affect only a_t but not g_t. Both G_t and g_t are computed from information already know before any observation on $(Z_1, ..., Z_T)$ is made. The optimal policy X_t is set by (38), using G_t and g_t and the observation on Z_{t-1}

The Steady State Solution

- Under which circumstances are the results for G_t and g_t stable over time?
- Look at (35) and (36)

$$G_t = -(C_t' H_t C_t)^{-1} C_t' H_t B_t \tag{35}$$

$$H_t = K_t + (B_{t+1} + C_{t+1} G_{t+1})' H_{t+1} (B_{t+1} + C_{t+1} G_{t+1}) \tag{36}$$

- Clearly, B_t, C_t and K_t must be invariant over time
- Then

$$G = -(C' HC)^{-1} C' HB \tag{39}$$

$$\begin{aligned} H &= K + (B + CG)' H (B + CG) \\ &= K + R'HR \end{aligned} \tag{40}$$

Finding R

- If a matrix can be found to make $R = (B + CG)$ so small that (40) can be satisfied for some H, then a steady-state solution for G_t exists.
- It can be shown that if the roots of $R = (B + CG)$ are all smaller than one, a steady state solution exists

Steady State Solution for g_t

- Analysing a steady state solution for g_t works in a similar way.
- Assume that B_t, C_t, b_t and K_t are all invariant through time. Equation (23) becomes

$$h_{t-1} = Ka_t + (B - CG_t)'(h_t - H_t b) \tag{41}$$

- So, h_t reaches a steady state when a_t, G_t and H_t are in steady state

Steady State Solution for g_t

- Remember, when G_t and H_t are in steady state, all roots of R are smaller than one
- The infinite series

$$I + R' + R'^2 + R'^3 + \ldots$$

will converge to $(I - R')^{-1}$ or the matrix $(I - R')$ will be non-singular

- The steady state form can be found by solving

$$h = Ka + R'(h - Hb)$$

$$(I - R')h = Ka - R'Hb$$

$$h = (I - B - CG)^{-1}(Ka - R'Hb) \qquad (42)$$

- Steady state solution for G_t and g_t and the optimal feedback equation if $B,\ C,\ b,\ K,$ and a are time-invariant

Dynamic Programming

- Dynamic Programming was suggested in this form by Richard Bellman in 1957
- Recall the linear model of the previous section

$$Z_t = B_t Z_{t-1} + C_t X_t + b_t + u_t \tag{43}$$

- and the quadratic loss function

$$W = \sum_{t=1}^{T} (Z_t - a_t)' K_t (Z_t - a_t) = \sum_{t=1}^{T} (Z_t' K_t Z_t - 2 Z_t' K_t a_t + a_t' K_t a_t) \tag{44}$$

The Method of Dynamic Programming

- The problem is to choose the optimal policy $X_1, \ldots, X_T$ which minimises the conditional expectation $E_0(W)$, given the initial condition Z_0
- By the method of **dynamic programming** the problem is solved first for the last period T, given the initial condition Z_{T-1}. Having found the optimal policy X_T for the last period, the two-period problem is solved for the last two periods by finding the optimal X_{T-1}, given the initial condition Z_{T-2}, et cetera. At the last stage the optimal X_1 for the first period is found, given the initial condition Z_0
- Thus, one problem with T unknowns is transformed into T problems with only one unknown each

Principle of Optimality

- By the principle of optimality the solution by dynamic programming is optimal because at each time t, whatever the initial condition, all future policies are optimal

Solution to last period

- The problem for the last period is

$$V_T = E_{T-1}(Z_T - a_T)'K_T(Z_T - a_T) = E_{T-1}(Z_T{}'H_T Z_T - 2Z_T{}'h_T + c_T) \quad (45)$$

  where $H_T = K_T, h_T = k_T \equiv K_T a_T,$ and $c_T = a_T' K_T a_T$

- Use model (43) for Z_T and take expectations. Note that the stochastic disturbance u_T is independent of Z_{T-1}
- Minimise

$$\begin{aligned} V_T = {} & (B_T Z_{T-1} + C_T X_T + b_T)' H_T (B_T Z_{T-1} + C_T X_T + b_T) \\ & -2(B_T Z_{T-1} + C_T X_T + b_T)' h_T + E_{T-1}(u_T' H_T u_T) + c_T \end{aligned} \quad (46)$$

- Differentiating with respect to X_T

$$\frac{\delta V_T}{\delta X_T} = 2C_T' H_T (B_T Z_{T-1} + C_T X_T + b_T) - 2C_T' h_T = 0 \quad (47)$$

Solution to last period (cont'd)

- The solution of (47) gives the optimal policy for the last period

$$\hat{X}_T = G_T Z_{T-1} + g_T \tag{48}$$

where

$$G_T = -(C_T' H_T C_T)^{-1}(C_T' H_T B_T) \tag{49}$$

$$g_T = -(C_T' H_T C_T)^{-1}(C_T' H_T b_T - C_T' h_T) \tag{50}$$

- Consider the problem for period $T-1$
- The two-period problem is to find the optimal policies X_T and X_{T-1}
- But, solution for X_T is already given for the last period as a function of Z_{T-1}
- The only remaining problem is to choose the optimal X_{T-1}
- The choice of X_{T-1} will affect Z_{T-1}, but whatever Z_{T-1} will be, the X_T found before will always be optimal
- This logic carries over to any period

Minimum expected welfare loss in T

- To obtain the minimum expected welfare loss for the last period, conditional on the data Z_{T-1}, we substitute (48) for X_T in (46)

$$
\begin{aligned}
\hat{V}_T = {} & Z'_{T-1}(A_T + C_T G_T)' H_T (A_T + C_T G_T) Z_{T-1} \\
& + 2Z'_{T-1}(A_T + C_T G_T)'(H_T b_T - h_T) \\
& + (b_T + C_T g_T)' H_T (b_T + C_T g_T) - 2(b_T + C_T g_T)' h_T \\
& + c_T + E_{T-1} u'_T H_T u_T
\end{aligned} \tag{51}
$$

Solution to period $T-1$

- The following needs to be minimised with respect to X_{T-1}

$$V_{T-1} = E_{T-2}(Z'_{T-1}K_{T-1}Z_{T-1} - 2Z'_{T-1}k_{T-1} + a'_{T-1}K_{T-1}a_{T-1} + \hat{V}_T) \tag{52}$$

- Substitute (51) for $\hat{V}_T$ into (52)

$$V_{T-1} = E_{T-2}(Z'_{T-1}H_{T-1}Z_{T-1} - 2Z'_{T-1}h_{T-1} + c_{T-1}) \tag{53}$$

where

$$H_{T-1} = K_{T-1} + (A_T + C_T G_T)' H_T (A_T + C_T G_T) \tag{54}$$

$$h_{T-1} = k_{T-1} + (A_T + C_T G_T)'(h_T - H_T b_T) \tag{55}$$

$$c_{T-1} = a'_{T-1}K_{T-1}a_{T-1} + (b_T + C_T g_T)' H_T (b_T + C_T g_T) \tag{56}$$

$$-2(b_T + C_T g_T)' h_T + c_T + E_{T-1} u'_T H_T u_T$$

Complete solution

- Note that (53) has the same form as (45)
- Thus, the process from (45) to (56) can be repeated with the subscripts replaced accordingly
- This is the complete solution

- **Note that this solution is identical to the solution found using Lagrange multipliers!**