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Empirical Macroeconomics

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Topic 3: Models of Rational Expectations

- Introduction
- Solving Rational Expectations
- Example: Cagan Model
- Example: Cagan Model with too many stable roots

Introduction

- What are rational expectations?
- As John F. Muth wrote in 1961

> « The hypothesis asserts three things: (1) Information is scarce, and the economic system generally does not waste it. (2) The way expectations are formed depends specifically on the structure of the relevant system describing the economy. (3) A "public prediction", in the sense of Grunberg and Modigliani (1954), will have no substantial effect on the operation of the economic system (unless it is based on inside information). »

Solving Rational Expectations

- A linear model can be represented in state-space form as follows

$$\begin{bmatrix} x_{1,t} \\ E_t x_{2,t+1} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \qquad (1)$$

where $x_{1,t}$ is an n_1-vector of predetermined values with initial values given, $x_{2,t}$ is an n_2-vector of “forward looking” variables, and ε_t is white noise with covariance Σ

Solving Rational Expectations

- Take expectation of (1)

$$E_t \begin{bmatrix} x_{1,t} \\ x_{2,t+1} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} \tag{2}$$

- Calculate the Schur decomposition of A in (1) and reorder both T and Z so that the eigenvalues with modulus smaller than one come first
- If there are n_θ stable (i.e. in modulus smaller than one) eigenvalues and n_δ unstable eigenvalues, T can be partitioned as

$$T = \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ 0 & T_{\delta\delta} \end{bmatrix}$$

Auxiliary variables and Schur decomposition

- Introduce the auxiliary variables

$$\begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} = Z^H \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \tag{3}$$

- According to the Schur decomposition, (2) can be written as

$$E_t \begin{bmatrix} x_{1,t} \\ x_{2,t+1} \end{bmatrix} = ZTZ^H \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

- Pre-multiplying with Z^H and using (3) gives

$$E_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} = Z^H ZT \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}$$

$$= \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ 0 & T_{\delta\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \tag{4}$$

Simplify

- Because $T_{\delta\delta}$ contains the unstable eigenvalues, δ_t diverges when t increases unless $\delta_0 = 0$. Any stable system requires therefore that $\delta_t = 0$ for all t. Thus (4) can be simplified to

$$\begin{aligned} \delta_t &= 0 \\ E_t\theta_{t+1} &= T_{\theta\theta}\theta_t \end{aligned} \qquad (5)$$

- Invert (3) and partition as

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} Z_{k\theta} & Z_{k\delta} \\ Z_{\lambda\theta} & Z_{\lambda\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} = \begin{bmatrix} Z_{k\theta} \\ Z_{\lambda\theta} \end{bmatrix} \theta_t \qquad (6)$$

since $\delta_t = 0$

Proposition 1 of Blanchard & Kahn (1980)

- The initial condition is that $x_{1,0}$ is given
- From (6)

$$x_{10} = Z_{k\theta}\theta_0 \tag{7}$$

which can be solved for θ_0 if $Z_{k\theta}$ is invertible. The latter has n_1 rows (the number of predetermined/backward looking variables) and n_δ columns (as many as stable roots)

- It follows that the number of predetermined variables must be equal to the number of stable roots (Prop. 1 B&K1980)
- Then

$$\theta_0 = Z_{k\theta}^{-1}x_{10} \tag{8}$$

so the stable solution can be calculated using (8) and (4), and then transforming back to $x_{1,t}$

Complete solution

- From (1) and (2) we know that $x_{1,t+1} - E_t x_{1,t+1} = \varepsilon_{t+1}$. Using (6) the latter may be written as

$$Z_{k\theta}\left(\theta_{t+1} - E_t\theta_{t+1}\right) = \varepsilon_{t+1} \tag{9}$$

- Under the same conditions as mentioned above for $Z_{k\theta}$, this can be inverted and written as

$$\theta_{t+1} = E_t\theta_{t+1} + Z_{k\theta}^{-1}\varepsilon_{t+1} \tag{10}$$

- Combine with (5) results in

$$\theta_{t+1} = T_{\theta\theta}\theta_t + Z_{k\theta}^{-1}\varepsilon_{t+1} \tag{11}$$

which is, together with (8) and (6) a complete solution of the stochastic model

Re-introduce dynamics

- However, we need to re-introduce the dynamics of the system
- Using $\theta_t = Z_{k\theta}^{-1} x_{1,t}$ from (6) in (11) gives

$$x_{1,t+1} = Z_{k\theta} T_{\theta\theta} Z_{k\theta}^{-1} x_{1,t} + \varepsilon_{t+1} \tag{12}$$

- Similarly, combining $x_{2,t} = Z_{\lambda\theta}\theta_t$ with $\theta_t = Z_{k\theta}^{-1} x_{1,t}$ (both from (6)) gives

$$x_{2,t} = Z_{\lambda\theta} Z_{k\theta}^{-1} x_{1,t} \tag{13}$$

- Summarise the last two equations to obtain

$$x_{1,t+1} = M x_{1,t} + \varepsilon_{t+1} \text{ , and} \tag{14}$$

$$x_{2,t} = C x_{1,t} \tag{15}$$

where the definitions of M and C are obvious, looking at (12) and (13).

Example: Cagan Model

- Given

$$\ln M_t - \ln P_t = -\omega i_t \quad \text{where} \quad \omega > 0 \qquad \text{(E1)}$$

where M_t, P_t and i_t denote the real money balance, the price level and the nominal interest rates

- The real interest rate is assumed to be invariant over time, so the Fisher equation becomes

$$i_t = E_t(\ln P_{t+1} - \ln P_t) + cons \qquad \text{(E2)}$$

- Combining the two equations and rearranging gives

$$\ln P_t = (1-\alpha)\ln M_t + \alpha E_t P_{t+1}, \quad \text{with} \quad 0 < \alpha < 1 \text{ as } \alpha = \omega/(1+\omega) \qquad \text{(E3)}$$

Model in state-space form

- Assume that the money supply is an exogenous AR(1)

$$\ln M_{t+1} = \rho \ln M_t + \varepsilon_{t+1} \qquad \text{(E4)}$$

- The model can be rewritten and represented in a state-space form as in (1)

$$\begin{bmatrix} \ln M_{t+1} \\ E_t \ln P_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ \frac{\alpha-1}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \ln M_t \\ \ln P_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \qquad \text{(E5)}$$

- The eigenvalues of the matrix A are ρ and $1/\alpha$. Now, let's assign $\alpha = 0.5$ and $\rho = 0.9$
- The Schur decomposition is then

$$A = \begin{bmatrix} \rho & 0 \\ \frac{\alpha-1}{\alpha} & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ -1 & 2 \end{bmatrix}$$

and

$$Z \approx \begin{bmatrix} -0.740 & -0.673 \\ -0.673 & 0.740 \end{bmatrix},\ T = \begin{bmatrix} 0.9 & -1 \\ 0 & 2 \end{bmatrix},\ Z^H \approx \begin{bmatrix} -0.740 & -0.673 \\ -0.673 & 0.740 \end{bmatrix} \quad \text{(E6)}$$

Solving

- To solve the model in (E5) recall equation (13) and write

$$\begin{aligned}\ln P_t &= Z_{21}Z_{11}^{-1}\ln M_t \\ &= -0.673\cdot(-.74)^{-1}\ln M_t \\ &\approx 0.909\ln M_t\end{aligned}$$

- Equation (12) gives the solution for the AR(1) process of money supply

$$\begin{aligned}\ln M_{t+1} &= Z_{11}T_{11}Z_{11}^{-1}\ln M_t + \varepsilon_{t+1} \\ &= -0.74\cdot 0.9\cdot(-0.74)^{-1}\ln M_t + \varepsilon_{t+1} \\ &= 0.9\ln M_t + \varepsilon_{t+1}\end{aligned}$$

Example: Cagan Model with too many stable roots

- Consider again the former example but change the price equation to

$$\ln P_t = \ln M_t + aE_t P_{t+1}, \quad \text{with} \quad 0 < a < 1 \tag{E7}$$

- The model can be written in matrix notation as

$$\begin{bmatrix} \ln M_{t+1} \\ E_t \ln P_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\frac{1}{a} & \frac{1}{a} \end{bmatrix} \begin{bmatrix} \ln M_t \\ \ln P_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \tag{E8}$$

with the eigenvalues ρ and $1/\alpha$

Fundamental Solution

- For illustration, suppose that $|a\,\rho| < 1$
- Then, iterating on the price equation in (E7) gives the stable fundamental solution

$$\ln P_t^* = \sum_{s=0}^{\infty} a^s E_t \ln M_{t+s}$$

$$= \frac{1}{1 - a\rho} \ln M_t$$

Bubble

- However, the full set of solutions is $\ln P_t = \ln P_t^* + b_t$, where b_t is a bubble
- Try this in (E7) to get

$$\frac{1}{1-a\rho}\ln M_t + b_t = \ln M_t + aE_t\left(\frac{1}{1-a\rho}\ln M_{t+1} + b_{t+1}\right)$$

$$\frac{1}{1-a\rho}\ln M_t + b_t = \ln M_t + \frac{a\rho}{1-a\rho}\ln M_t + aE_t b_{t+1}$$

- It can be seen that the term simplifies to $b_t = aE_t b_{t+1}$
- What are the consequences?

Bubble (cont'd)

- This means that $E_t b_{t+1} = b_t / a$
- When $|a| < 1$ the bubble is unstable and we may choose $b_t = 0$ to obtain an economically meaningful (i.e. stable) solution of the price level $\ln P_t = \ln P_t * + b_t$
- However, with $|a| > 1$ there is an infinite number of stable bubbles, which all have a stable price level)
- There is no reason to choose one over the other!